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Four dimensional "old minimal" $\mathcal{N}=2$ supersymmetrization of $\mathcal{R}^4$

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Abstract: We write in superspace the lagrangian containing the fourth power of the Weyl tensor in the "old minimal" $d=4$, $\mathcal{N}=2$ supergravity, without local SO(2) symmetry. Using gauge completion, we analyze the lagrangian in components. We find out that the auxiliary fields which belong to the Weyl and compensating vector multiplets have derivative terms and therefore cannot be eliminated on-shell. Only the auxiliary fields which belong to the compensating nonlinear multiplet do not get derivatives and could still be eliminated; we check that this is possible in the leading terms of the lagrangian. We compare this result to the similar one of "old minimal" $\mathcal{N}=1$ supergravity and we comment on possible generalizations to other versions of $\mathcal{N}=1,2$ supergravity.

Keywords: Supergravity Models, Supersymmetric Effective Theories, Superspaces.

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1. Introduction

The supersymmetrization of the fourth power of the Riemann tensor has been an active topic of research. In four dimensions, a term like this would be the leading bosonic contribution to a possible three-loop supergravity counterterm [1]. In type I supergravity in ten dimensions, an $\mathcal{R}^4$ term is necessary to cancel gravitational anomalies [2, 3]. $\mathcal{R}^4$ terms also show up in the low energy field theory effective action of both type II and heterotic string theories, as was shown in [4]–[6].

In previous papers [7, 8] we have worked out the $\mathcal{N}=1$ supersymmetrization of $\mathcal{R}^4$ in four dimensions. We have shown that, in the "old minimal" formulation with this term, the auxiliary fields M, N could still be eliminated, but A_m could not (it got derivative couplings in the lagrangian that led to a dynamical field equation).

The goal of this article is to extend this supersymmetrization to $\mathcal{N}=2$ supergravity, which also admits off-shell formulations, and compare the result to the $\mathcal{N}=1$ one.

We start by briefly reviewing how one can obtain the "old minimal" (without local SO(2)) $\mathcal{N}=2$ Poincaré supergravity from the conformal theory by coupling to compensating vector and nonlinear multiplets. We then write, in superspace, using the known chiral projector and chiral density, a lagrangian that contains the fourth power of the Weyl tensor. We start expanding this action in components and we quickly conclude that some of these auxiliary fields get derivatives and cannot be eliminated. For the other auxiliary fields, we make a more detailed analysis and we show that their derivative terms cancel and, therefore, it should still be possible to eliminate them. We analyze, for which multiplet (Weyl, compensating vector and compensating nonlinear) which auxiliary fields can and which cannot be eliminated. We proceed analogously in the $\mathcal{N}=1$ case, using the previous known results. We compare the two cases and discuss what can be generalized to other versions of both these theories.

In appendix B we give a survey of curved SU(2) superspace, namely its field content and the solution of the Bianchi identities.

2. $\mathcal{N}=2$ supergravity in superspace

2.1 $\mathcal{N}=2$ conformal supergravity in superspace

Conformal supergravity theories were found for $\mathcal{N} \leq 4$ in four dimensions. These theories have a local internal U($\mathcal{N}$) symmetry which acts on the supersymmetry generators Q^a_A and S^a_A, with $a = 1, \ldots, \mathcal{N}$; they can be formulated off-shell in conventional extended superspace, with structure group $\mathrm{SO}(1,3) \times \mathrm{U}(\mathcal{N})$, where their actions, written as chiral superspace integrals, are known at the full nonlinear level. The first discussion of the conformal properties of extended superspace was given in [9]; other later references are the seminal paper [10] and the very nice review [11]. Here we summarize the main results we need.

In superspace, the main objects are the supervielbein E^M_Π and the superconnection ${\Omega_{\Lambda N}}^P$ (which can be decomposed in its Lorentz and U($\mathcal{N}$) parts), in terms of which we write the torsions $T_{MN}{}^{P}$ and curvatures $R_{MN}{}^{PQ}$. Their arbitrary variations are given by

$$\begin{aligned} H_M{}^N &= E_M{}^\Lambda \delta E_\Lambda{}^N \\ \Phi_{MN}{}^P &= E_M{}^\Lambda \delta\Omega_{\Lambda N}{}^P . \end{aligned} \tag{2.1}$$

Symmetries that are manifest in superspace are general supercoordinate transformations (which include x-space diffeomorphisms and local supersymmetry), with parameters ξ^Λ, and tangent space (structure group) transformations, with parameters Λ^{MN}. One can solve for $H_M{}^N$ and $\Phi_{MN}{}^P$ in terms of these parameters, torsions and curvatures as

$$\begin{aligned} H_M^N &= \xi^P T_{PM}{}^N + \nabla_M \xi^N + \Lambda_M^N \\ \Phi_{MN}{}^P &= \xi^Q R_{QMN}{}^P - \nabla_M \Lambda_N^P , \end{aligned} \tag{2.2}$$

but this does not fix all the degrees of freedom of $H_M{}^N$ [10, 11]. Namely, $H = -\frac{1}{4}H_m^m$ remains an unconstrained superfield and parametrizes the super-Weyl transformations, which include the dilatations and the special supersymmetry transformations.

$\mathcal{N} = 1, 2$ Poincaré supergravities can be obtained from the corresponding conformal theories by consistent couplings to compensating multiplets that break superconformal invariance and local U($\mathcal{N}$). There are different possible choices of compensating multiplets, leading to different formulations of the Poincaré theory. Because of its relevance to this paper, we will briefly review the $\mathcal{N} = 2$ case, which was first studied in [9].

The $\mathcal{N} = 2$ Weyl multiplet has 24+24 degrees of freedom. Its field content is given by the graviton e_μ^m, the gravitinos ψ_μ^{Aa}, the U(2) connection $\widetilde{\Phi}_\mu^{ab}$, an antisymmetric tensor W_{mn} which we decompose as $W_{A\dot{A}B\dot{B}} = 2\varepsilon_{\dot{A}\dot{B}} W_{AB} + 2\varepsilon_{AB} W_{\dot{A}\dot{B}}$, a spinor Λ_A^a and, as auxiliary field, a dimension 2 scalar I. In superspace, a gauge choice can be made (in the supercoordinate transformation) such that the graviton and the gravitinos are related to $\theta = 0$ components of the supervielbein (symbolically $E_\Pi{}^N|$):

$$E_\Pi{}^N\big| = \begin{bmatrix} e_\mu{}^m & \frac{1}{2}\psi_\mu{}^{Aa} & \frac{1}{2}\psi_\mu{}^{\dot{A}a} \\ 0 & -\delta_B{}^A \delta_b{}^a & 0 \\ 0 & 0 & -\delta_{\dot{B}}{}^{\dot{A}} \delta_b{}^a \end{bmatrix} . \tag{2.3}$$

In the same way, we gauge the fermionic part of the Lorentz superconnection at order $\theta = 0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\begin{aligned} \Omega_{\mu m}{}^n\big| &= \omega_{\mu m}{}^n(x) \\ \Omega_{Aam}{}^n\big| , \Omega_{\dot{A}am}{}^n\big| &= 0 . \end{aligned} \tag{2.4}$$

The U(2) superconnection $\widetilde{\Phi}_\Pi^{ab}$ is such that

$$\widetilde{\Phi}_\mu^{ab}\Big| = \widetilde{\Phi}_\mu^{ab} . \tag{2.5}$$

The other fields are the $\theta = 0$ component of some superfield, which we write in the same way.

The chiral superfield W_{AB} is the basic object of $\mathcal{N} = 2$ conformal supergravity, in terms of which its action is written. Other theories with different $\mathcal{N}$ have its analogous superfield (e.g. W_{ABC} in $\mathcal{N}=1$), but with different spinor and (S)U($\mathcal{N}$) indices. A common feature to these superfields is having the antiself-dual part of the Weyl tensor $\mathcal{W}_{ABCD} := -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}$ in their θ expansion.

In U(2) $\mathcal{N}=2$ superspace there is an off-shell solution to the Bianchi identities. The torsions and curvatures can be expressed in terms of superfields W_{AB}, Y_{AB}, $U^{ab}_{A\dot{A}}$, X_{ab}, their complex conjugates and their covariant derivatives. Of these four superfields, only W_{AB} transforms covariantly under super-Weyl transformations [10]:

$$\delta W_{AB} = H W_{AB} \,. \tag{2.6}$$

The other three superfields transform non-covariantly; they describe all the non-Weyl covariant degrees of freedom in H, and can be gauged away by a convenient (Wess-Zumino) gauge choice.

$$\begin{aligned}
\delta Y_{AB} &= H Y_{AB} - \frac{1}{4}\left[\nabla^a_A, \nabla_{Ba}\right] H \\
\delta X_{ab} &= H X_{ab} + \frac{1}{4}\left[\nabla^{\dot{A}}_a, \nabla_{\dot{A}b}\right] H \\
\delta U^{ab}_{A\dot{A}} &= H U^{ab}_{A\dot{A}} - \frac{1}{2}\left[\nabla^a_A, \nabla^b_{\dot{A}}\right] H \,.
\end{aligned} \tag{2.7}$$

Another nice feature of $\mathcal{N}=2$ superspace is that there exists a chiral density ϵ and an antichiral projector, given by [11]

$$\nabla^{Aa}\nabla^b_A\left(\nabla^B_a\nabla_{Bb} + 16X_{ab}\right) - \nabla^{Aa}\nabla^B_a\left(\nabla^b_A\nabla_{Bb} - 16iY_{AB}\right). \tag{2.8}$$

When one acts with this projector on any scalar superfield, one gets an antichiral superfield (with the exception of W_{AB}, only scalar chiral superfields exist in curved $\mathcal{N}=2$ superspace). It is then possible to write chiral actions [12].

2.2 Degauging U(1)

The first step for obtaining the Poincaré theory is to couple to the conformal theory an abelian vector multiplet (with central charge), described by a vector A_μ, a complex scalar, a Lorentz-scalar SU(2) triplet and a spinorial SU(2) dublet. The vector A_μ is the gauge field of central charge transformations; it corresponds, in superspace, to a 1-form A_Π with a U(1) gauge invariance (the central charge transformation). This 1-form does not belong to the superspace geometry.

Using the U(1) gauge invariance we can set the gauge

$$A_\Pi| = (A_\mu, 0) \,. \tag{2.9}$$

The field strength $F_{\Pi\Sigma}$ is a two-form satisfying its own Bianchi identities $\nabla_{[\Gamma}F_{\Pi\Sigma\}} = 0$. Here we split the U(2) superconnection $\widetilde{\Phi}^{ab}_\Pi$ into a SU(2) superconnection Φ^{ab}_Π and a U(1) superconnection φ_Π; only the later acts on A_Π:

$$\widetilde{\Phi}^{ab}_\Pi = \Phi^{ab}_\Pi - \frac{1}{2}\varepsilon^{ab}\varphi_\Pi \,. \tag{2.10}$$

One has to impose covariant constraints on its components (like in the torsions), in order to construct invariant actions:

$$
\begin{aligned}
F^{ab}_{AB} &= 2\sqrt{2}\varepsilon_{AB}\varepsilon^{ab}F \\
F^{ab}_{A\dot{B}} &= 0 \,.
\end{aligned}
\tag{2.11}
$$

By solving the $F_{\Pi\Sigma}$ Bianchi identities with these constraints, we conclude that they define an off-shell $\mathcal{N}=2$ vector multiplet, given by the $\theta=0$ components of the superfields

$$
A_\mu, F, F^a_A = \frac{i}{2}F^{\dot{A}a}{}_{A\dot{A}}, F^a_b = \frac{1}{2}\left(-\nabla^B_b F^a_B + F\overline{X}^a{}_b + \overline{F}X^a{}_b\right)
\tag{2.12}
$$

$F^a_b|$ is an auxiliary field; $F^a_a = 0$ if the multiplet is abelian (as it has to be in this context). $\overline{F}$ is a Weyl covariant chiral superfield, with nonzero U(1) and Weyl weigths. A superconformal chiral lagrangian for the vector multiplet is given by

$$
\mathcal{L} = \int \bar{\epsilon}F^2 d^4\overline{\theta} + \text{h.c.}\,.
\tag{2.13}
$$

In order to get a Poincaré theory, we must break the superconformal and local abelian (from the U(1) subgroup of U(2) — not the gauge invariance of A_μ) invariances. For that, we set the Poincaré gauge

$$
F = \overline{F} = 1\,.
\tag{2.14}
$$

As a consequence, from the Bianchi and Ricci identities we get

$$
\begin{aligned}
\varphi^a_A &= 0 \\
F^A_a &= 0\,.
\end{aligned}
\tag{2.15}
$$

Furthermore, $U^{ab}_{A\dot{A}}$ is an SU(2) singlet, to be identified with the bosonic U(1) connection (now an auxiliary field):

$$
U^{ab}_{A\dot{A}} = \varepsilon^{ab}U_{A\dot{A}} = \varepsilon^{ab}\varphi_{A\dot{A}}\,.
\tag{2.16}
$$

Other consequences are

$$
F_{A\dot{A}B\dot{B}} = \sqrt{2}i\left[\varepsilon_{AB}\left(W_{\dot{A}\dot{B}} + Y_{\dot{A}\dot{B}}\right) + \varepsilon_{\dot{A}\dot{B}}\left(W_{AB} + Y_{AB}\right)\right]
\tag{2.17}
$$

$$
F^a_b = X^a_b
\tag{2.18}
$$

$$
\overline{X_{ab}} = X^{ab}
\tag{2.19}
$$

(2.17) shows that W_{mn} is now related to the vector field strength F_{mn}. Y_{mn} emerges as an auxiliary field, like X_{ab} (from (2.18)). We have, therefore, the minimal field representation of $\mathcal{N}=2$ Poincaré supergravity, with a local SU(2) gauge symmetry and 32+32 off-shell degrees of freedom:

$$
e^m_\mu, \psi^{Aa}_\mu, A_\mu, \Phi^{ab}_\mu, Y_{mn}, U_m, \Lambda^a_A, X_{ab}, I\,.
\tag{2.20}
$$

Although the algebra closes with this multiplet, it does not admit a consistent lagrangian because of the higher-dimensional scalar I [14].

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2.3 Degauging SU(2)

The second step is to break the remaining local SU(2) invariance. This symmetry can be partially broken (at most, to local SO(2)) through coupling to a compensating so-called "improved tensor multiplet" [15, 16], or broken completely. In this work, we take the later possibility. There are still two different versions of off-shell $\mathcal{N}=2$ supergravity without SO(2) symmetry, each with different physical degrees of freedom. In both cases we start by imposing a constraint on the SU(2) parameter L^{ab} which restricts it to a compensating nonlinear multiplet [17]:[1]

$$\nabla_A^{\underline{a}} L^{\underline{bc}} = 0\,. \tag{2.21}$$

From the transformation law of the SU(2) connection

$$\delta\Phi_M^{ab} = -\nabla_M L^{ab} \tag{2.22}$$

we can get the required condition for L^{ab} by imposing the following constraint on the fermionic connection:

$$\Phi_A^{abc} = 2\varepsilon^{a\underline{b}}\rho_A^{\underline{c}}\,. \tag{2.23}$$

This constraint requires introducing a new fermionic superfield ρ_A^a. We also introduce its fermionic derivatives P (a complex scalar) and H_m (see appendix B.1). The previous SU(2) connection Φ_μ^{ab} is now an unconstrained auxiliary field. The divergence of the vector field H_m is constrained, though, at the linearized level by the condition $\nabla^m H_m = \frac{1}{3}R - \frac{1}{12}I$. The full nonlinear constraint is

$$\begin{aligned} I = {} & 4R - 6\nabla_{A\dot{A}}H^{A\dot{A}} - 24X^{ab}X_{ab} - 12W^{AB}Y_{AB} - 12W^{\dot{A}\dot{B}}Y_{\dot{A}\dot{B}} + \\ & + 3P\overline{P} + \frac{3}{2}H^{A\dot{A}}H_{A\dot{A}} - 12\Phi_{ab}^{A\dot{A}}\Phi_{A\dot{A}}^{ab} - 12U^{A\dot{A}}U_{A\dot{A}} + 16i\rho_A^a\Lambda_a^A - \\ & - 16i\rho_{\dot{A}}^a\Lambda_a^{\dot{A}} - 48\rho_a^A W_{AB}{}^{Ba} + 48\rho_a^{\dot{A}}W_{\dot{A}\dot{B}}{}^{\dot{B}a} + 48i\rho_a^A\rho^{Ba}W_{AB} + \\ & + 48i\rho_a^{\dot{A}}\rho^{\dot{B}a}W_{\dot{A}\dot{B}} + 48\rho^{Aa}\rho_a^{\dot{A}}U_{A\dot{A}} - 48i\rho^{Aa}\nabla_{A\dot{A}}\rho_a^{\dot{A}} + \\ & + 48i\rho^{\dot{A}a}\nabla_{A\dot{A}}\rho_a^A + 96i\rho_a^A\Phi_{A\dot{A}}^{ab}\rho_b^{\dot{A}} \end{aligned} \tag{2.24}$$

which is equivalent to saying that I, now defined by (B.10), is no longer an independent field. This constraint implies that only the longitudinal part of H_m belongs to the nonlinear multiplet; its divergence lies in the original Weyl multiplet. From the structure equation

$$R_{MN}{}^{ab} = E_M{}^\Lambda E_N{}^\Pi \left\{\partial_\Lambda\Omega_\Pi^{ab} + \Omega_\Lambda{}^{ac}\Omega_{\Pi c}{}^b - (-)^{\Lambda\Pi}(\Lambda\leftrightarrow\Pi)\right\} \tag{2.25}$$

and the constraint/definition (2.23), we can derive off-shell relations for the (still SU(2) covariant) derivatives of ρ_A^a, which we collect in appendix B.2.

Altogether, these component fields form then the "old minimal" $\mathcal{N}=2$ 40+40 multiplet [18]:

$$e_\mu^m, \psi_\mu^{Aa}, A_\mu, \Phi_\mu^{ab}, Y_{mn}, U_m, \Lambda_A^a, X_{ab}, H_m, P, \rho_A^a\,. \tag{2.26}$$

[1] Actually, condition (2.21) restricts L^{ab} to a tensor multiplet, which is the linearization of the nonlinear multiplet. This is enough for our analysis.

This is the formulation of $\mathcal{N}=2$ supergravity we are working with. The other possibility (also with SU(2) completely broken) is to further restrict the compensating non-linear multiplet to an on-shell scalar multiplet [19]. This reduction generates a minimal 32+32 multiplet (not to be confused with (2.20)) with new physical degrees of freedom. We will not pursue this version of $\mathcal{N}=2$ supergravity in this work.

2.4 $\mathcal{N}=2$ Poincaré supergravity in superspace

The final lagrangian of "old minimal" $\mathcal{N}=2$ supergravity is given by[2] [17, 20]

$$\begin{aligned}
\kappa^2\mathcal{L}_{SG} = & -\frac{1}{2}e\mathcal{R} - \frac{1}{4}e\varepsilon^{\mu\nu\rho\lambda}\left(\psi_{\mu\dot{A}}^{\ a}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda Aa} + \psi_{\mu A}^{\ a}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda\dot{A}a}\right) - \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} - \\
& - \frac{1}{4\sqrt{2}}e\left(\psi_\mu^{Aa}\psi_{\nu Aa} + \psi_\mu^{\dot{A}a}\psi_{\nu\dot{A}a}\right)F^{\mu\nu} - \frac{i}{4\sqrt{2}}e\varepsilon^{\mu\nu\rho\lambda}\left(\psi_\mu^{Aa}\psi_{\nu Aa} - \psi_\mu^{\dot{A}a}\psi_{\nu\dot{A}a}\right)F_{\rho\lambda} - \\
& - \frac{i}{16}e\varepsilon^{\mu\nu\rho\lambda}\left(\psi_\mu^{Aa}\psi_{\nu Aa} - \psi_\mu^{\dot{A}a}\psi_{\nu\dot{A}a}\right)\left(\psi_\rho^{Bb}\psi_{\lambda Bb} + \psi_\rho^{\dot{B}b}\psi_{\lambda\dot{B}b}\right) - \\
& - \frac{1}{4}eY_{\mu\nu}Y^{\mu\nu} - \frac{1}{2}eX^{ab}X_{ab} + eU^2 - \frac{1}{8}eP\overline{P} - \frac{1}{8}eH^2 + e\Phi^\mu_{ab}\Phi^{ab}_\mu + \frac{2}{3}ie\rho^{Aa}\Lambda_{Aa} - \\
& - \frac{2}{3}ie\rho^{\dot{A}a}\Lambda_{\dot{A}a} - 4e\rho^{Aa}W_{ABa}^{\ \ \ B} + 4e\rho^{\dot{A}a}W_{\dot{A}\dot{B}a}^{\ \ \ \dot{B}} + 2ie\rho^{Aa}\rho_a^B W_{AB} + 2ie\rho^{\dot{A}a}\rho_a^{\dot{B}}W_{\dot{A}\dot{B}} - \\
& - ieP\rho^{Aa}\rho_{Aa} - ie\overline{P}\rho^{\dot{A}a}\rho_{\dot{A}a} - 2e\rho^{Aa}\rho_a^{\dot{A}}U_{A\dot{A}} + \\
& + 2ie\rho^{Aa}\sigma^\mu_{A\dot{A}}\mathcal{D}_\mu\rho_a^{\dot{A}} - 2ie\rho^{\dot{A}a}\sigma^\mu_{A\dot{A}}\mathcal{D}_\mu\rho_a^A + ie\left(\rho^{Aa}\sigma^\mu_{A\dot{A}}\psi_\mu^{\dot{A}b} + \rho^{\dot{A}a}\sigma^\mu_{A\dot{A}}\psi_\mu^{Ab}\right)X_{ab} + \\
& + e\rho^{Aa}\sigma^\mu_{A\dot{A}}\psi_{\mu\dot{B}a}Y^{\dot{A}\dot{B}} + e\rho^{\dot{A}a}\sigma^\mu_{A\dot{A}}\psi_{\mu Ba}Y^{AB} + ie\rho^{Aa}\sigma^\mu_{A\dot{A}}\psi_{\mu Ba}U^{B\dot{A}} + \\
& + ie\rho^{\dot{A}a}\sigma^\mu_{A\dot{A}}\psi_{\mu\dot{B}a}U^{A\dot{B}} + \frac{1}{4}e\rho^{Aa}\sigma^\mu_{A\dot{A}}\psi_{\mu a}^{\ \ \dot{A}}P + \frac{1}{4}e\rho^{\dot{A}a}\sigma^\mu_{A\dot{A}}\psi_{\mu a}^{\ \ A}\overline{P} - \\
& - \left(\rho^{Aa}\psi_{\mu A}^{\ \ \ b} - \rho^{\dot{A}a}\psi_{\mu\dot{A}}^{\ \ \ b}\right)\left(\rho_b^B\sigma^{\mu\nu}_{BC}\psi^C_{\nu a} - \rho_b^{\dot{B}}\sigma^{\mu\nu}_{\dot{B}\dot{C}}\psi^{\dot{C}}_{\nu a}\right) - \rho_a^{\dot{A}}\sigma_\mu^{A\dot{B}}\psi^\mu_{\dot{B}b}\Phi^{ab}_{A\dot{A}} + \\
& + \rho_a^A\sigma_\mu^{B\dot{A}}\psi^\mu_{Bb}\Phi^{ab}_{A\dot{A}} - \frac{1}{4}e\rho^{Aa}\sigma_\mu^{B\dot{A}}\psi^\mu_{Ba}H_{A\dot{A}} + \frac{1}{4}e\rho^{\dot{A}a}\sigma_\mu^{A\dot{B}}\psi^\mu_{\dot{B}a}H_{A\dot{A}}\,.
\end{aligned} \tag{2.27}$$

The final solutions to the Bianchi identities in SU(2) $\mathcal{N}=2$ superspace are listed in appendix B. We present both the expressions for the torsions and curvatures and the off-shell differential relations among the superfields (appendix B.2). As first noticed in [9], these solutions only depend on W_{AB} (a physical field at $\theta=0$), ρ^a_A (an auxiliary field at $\theta=0$), their complex conjugates and their covariant derivatives. Here we present for completeness the full expansion of the (anti)chiral density $\overline{\epsilon}$ [20]:

$$\begin{aligned}
\overline{\epsilon} = & \ e - ie\theta_a^{\dot{A}}\sigma^\mu_{A\dot{A}}\psi_\mu^{Aa} + \\
& + \frac{e}{2}\theta_a^{\dot{A}}\theta_b^{\dot{B}}\left[\frac{16}{3}\varepsilon_{\dot{A}\dot{B}}X^{ab} + \frac{8}{3}i\varepsilon^{ab}Y_{\dot{A}\dot{B}} + \sigma^\mu_{A\dot{A}}\psi_\mu^{Aa}\sigma^\nu_{B\dot{B}}\psi_\nu^{Bb} - \sigma^\mu_{A\dot{A}}\psi_\nu^{Aa}\sigma^\nu_{B\dot{B}}\psi_\mu^{Bb}\right] -
\end{aligned}$$

[2] $\mathcal{D}_\mu$ is just the usual Lorentz covariant derivative (not U(2) covariant).

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$$
\begin{aligned}
&-\frac{e}{6}\theta_a^{\dot{A}}\theta_b^{\dot{B}}\theta_c^{\dot{C}}\left[\frac{16}{3}i\varepsilon_{\dot{A}\dot{B}}\varepsilon^{ac}\Lambda^b_{\dot{C}}+4\varepsilon_{\dot{A}\dot{B}}\varepsilon^{ac}\sigma^\mu_{C\dot{C}}\psi^b_{\mu B}Y^{BC}-\right.\\
&\qquad -i\varepsilon_{\dot{A}\dot{B}}\varepsilon^{ac}\sigma^\mu_{C\dot{C}}\sigma^{BC}_{\rho\lambda}\psi_{\mu B}^{\ b}\left(\sqrt{2}F^{\rho\lambda}+\psi^{\rho Ee}\psi^\lambda_{Ee}+\psi^{\rho\dot{E}e}\psi^\lambda_{\dot{E}e}\right)+\\
&\qquad +2\sigma^\mu_{A\dot{A}}\psi^{Aa}_\mu\left(7i\varepsilon_{\dot{B}\dot{C}}X^{bc}-5\varepsilon^{bc}Y_{\dot{B}\dot{C}}\right)+i\sigma^\mu_{A\dot{A}}\psi^{Aa}_\mu\sigma^\nu_{B\dot{B}}\psi^{Bb}_\nu\sigma^\lambda_{C\dot{C}}\psi^{Cc}_\lambda-\\
&\qquad \left.-3i\sigma^\mu_{A\dot{A}}\psi^{Aa}_\mu\sigma^\nu_{B\dot{B}}\psi^{Bb}_\lambda\sigma^\lambda_{C\dot{C}}\psi^{Cc}_\nu+2i\sigma^\mu_{A\dot{A}}\psi^{Aa}_\nu\sigma^\nu_{B\dot{B}}\psi^{Bb}_\lambda\sigma^\lambda_{C\dot{C}}\psi^{Cc}_\mu\right]-\\
&-\frac{2}{3}\theta_a^{\dot{A}}\theta_b^{\dot{B}}\theta^b_{\dot{A}}\theta^a_{\dot{B}}\left[\partial_\mu\left(-ieU^\mu+\frac{e}{2}H^\mu+2e\rho^A_a\sigma^{\mu\nu}_{AB}\psi^{Ba}_\nu-2e\rho^{\dot{A}}_a\sigma^{\mu\nu}_{\dot{A}\dot{B}}\psi^{\dot{B}a}_\nu\right)+\right.\\
&\qquad +\frac{e}{8}\varepsilon^{\mu\nu\rho\lambda}\left(iF_{\mu\nu}F_{\rho\lambda}-\psi_{\mu A}^{\ \ a}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda\dot{A}a}+\psi_{\mu\dot{A}}^{\ \ a}\sigma_\nu^{A\dot{A}}\psi_{\rho\lambda Aa}-\right.\\
&\qquad\qquad \left.\left.-i\psi^{Aa}_\mu\psi_{\nu Aa}\psi^{\dot{A}b}_\rho\psi_{\lambda\dot{A}b}\right)+\kappa^2\mathcal{L}_{SG}\right].
\end{aligned}
\tag{2.28}
$$

This allows us to write, up to total derivatives,

$$\mathcal{L}_{SG}=-\frac{3}{4\kappa^2}\int\bar{\epsilon}d^4\bar{\theta}+\text{h.c.}. \tag{2.29}$$

3. The supersymmetric $\mathcal{R}^4$ lagrangian

Our goal in this article is to supersymmetrize the fourth power of the Weyl tensor. (There are indeed, in four dimensions, thirteen independent scalar fourth-degree polynomials of the Riemann tensor, but we are only interested in a particular one. See [7] for a complete discussion.) As mentioned before, W_{AB} contains in its θ expansion the antiself-dual part of the Weyl tensor (see (B.21)).

3.1 The lagrangian in superspace

Analogously to [7], we write the supersymmetric $\mathcal{R}^4$ lagrangian in superspace, using the chiral projector and the chiral density, as a (quantum) correction to the pure supergravity lagrangian:

$$
\begin{aligned}
\mathcal{L}&=\int\bar{\epsilon}\left[-\frac{3}{4\kappa^2}+\alpha\kappa^4\left(\nabla^{Aa}\nabla^b_A\left(\nabla^B_a\nabla_{Bb}+16X_{ab}\right)-\right.\right.\\
&\qquad\left.\left.-\nabla^{Aa}\nabla^B_a\left(\nabla^b_A\nabla_{Bb}-16iY_{AB}\right)\right)W^2\overline{W}^2\right]d^4\bar{\theta}+\text{h.c.}\\
&=\mathcal{L}_{SG}+\mathcal{L}_{\mathcal{R}^4}
\end{aligned}
\tag{3.1}
$$

α is a (numerical) constant (we use a different definition from [7, 8]). Up to that (unknown) numerical factor, this can be seen as a three-loop $\mathcal{N}=2$ supergravity effective action or, equivalently, as a four dimensional $\mathcal{N}=2$ string/M-theory effective action resulting from a compactification and truncation from ten/eleven dimensions.

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After the superspace integration, (3.1) comes, in terms of components, as

$$\begin{aligned}\mathcal{L}_{\mathcal{R}^4} &= \alpha\kappa^4 \int \epsilon\phi d^4\theta + \text{h.c.} \\ &= \alpha\kappa^4 \Bigg(\epsilon^{(0)}\phi^{(4)} + \frac{1}{36}\epsilon^{(1)Bb}\phi^{(3)aA}_{BbAa} + \frac{1}{48}\epsilon^{(2)bAa}_{A}\phi^{(2)B}{}_{aBb} - \\ &\qquad - \frac{1}{48}\epsilon^{(2)}_{BaA}{}^{a}\phi^{(2)AbB}{}_{b} + \frac{1}{36}\phi^{(1)Bb}\epsilon^{(3)aA}_{BbAa} + \epsilon^{(4)}\phi^{(0)} \Bigg) + \text{h.c.}\,, \end{aligned} \tag{3.2}$$

where we have defined the chiral superfield

$$\phi = \left(\nabla^{\dot{A}a}\nabla^{b}_{\dot{A}}\left(\nabla^{\dot{B}}_{a}\nabla_{\dot{B}b} + 16X_{ab}\right) - \nabla^{\dot{A}a}\nabla^{\dot{B}}_{a}\left(\nabla^{b}_{\dot{A}}\nabla_{\dot{B}b} - 16iY_{\dot{A}\dot{B}}\right)\right) W^2\overline{W}^2 \tag{3.3}$$

and

$$\begin{aligned} \phi^{(0)} &= \phi| \\ \phi^{(1)Aa} &= \nabla^{Aa}\phi| \\ \phi^{(2)}_{AaBb} &= \frac{1}{2}\left[\nabla_{Aa}, \nabla_{Bb}\right]\phi| \\ \phi^{(3)}_{AaBbCc} &= \frac{1}{6}\left(\nabla_{Aa}\left[\nabla_{Bb}, \nabla_{Cc}\right] + \nabla_{Bb}\left[\nabla_{Cc}, \nabla_{Aa}\right] + \nabla_{Cc}\left[\nabla_{Aa}, \nabla_{Bb}\right]\right)\phi| \\ \phi^{(4)} &= \frac{1}{288}\,\nabla^{Aa}\nabla^{Bb}\left[\nabla_{Ab}, \nabla_{Ba}\right]\phi\Big|\,. \end{aligned} \tag{3.4}$$

As one can see from the component expansions in appendix B, the term $\epsilon^{(0)}\phi^{(4)} + \text{h.c.}$ clearly contains the fourth power of the Weyl tensor, more precisely (using the notation of [7]) $e\mathcal{W}_+^2\mathcal{W}_-^2$.

3.2 The lagrangian in components

We now proceed with the calculation of the components of ϕ and analysis of its field content. For that, we use the differential constraints from the solution to the Bianchi identities and the commutation relations listed in appendix B to compute the components in (3.4). The process is straightforward but lengthy.

We start by expanding ϕ as

$$\begin{aligned} \phi = {} & W^2\nabla^{\dot{A}a}\nabla^{b}_{\dot{A}}\nabla^{\dot{B}}_{a}\nabla_{\dot{B}b}\overline{W}^2 - W^2\nabla^{\dot{A}a}\nabla^{\dot{B}}_{a}\nabla^{b}_{\dot{A}}\nabla_{\dot{B}b}\overline{W}^2 - \\ & - 32iW^2\overline{W}^2\nabla^{\dot{A}a}\Lambda_{\dot{A}a} + 32iW^2\Lambda^{\dot{A}a}\nabla_{\dot{A}a}\overline{W}^2 - \\ & - 64iW^2\left(\nabla^{\dot{A}a}X_{a}{}^{b}\right)W^{\dot{B}\dot{C}}W_{\dot{B}\dot{C}\dot{A}b} - 64iW^2X_{a}{}^{b}\left(\nabla^{\dot{A}a}W^{\dot{B}\dot{C}}\right)W_{\dot{B}\dot{C}\dot{A}b} - \\ & - 64iW^2X_{a}{}^{b}W^{\dot{B}\dot{C}}\nabla^{\dot{A}a}W_{\dot{B}\dot{C}\dot{A}b} - \frac{64}{3}W^2\left(\nabla^{\dot{A}a}X_{ab}\right)W_{\dot{A}\dot{B}}\Lambda^{\dot{B}b} - \\ & - \frac{64}{3}W^2X_{ab}\left(\nabla^{\dot{A}a}W_{\dot{A}\dot{B}}\right)\Lambda^{\dot{B}b} - \frac{64}{3}W^2X_{ab}W_{\dot{A}\dot{B}}\nabla^{\dot{A}a}\Lambda^{\dot{B}b} - \\ & - 64W^2\left(\nabla^{a}_{\dot{A}}Y^{\dot{A}\dot{B}}\right)W^{\dot{C}\dot{D}}W_{\dot{C}\dot{D}\dot{B}a} - 64W^2Y^{\dot{A}\dot{B}}\left(\nabla^{a}_{\dot{A}}W^{\dot{C}\dot{D}}\right)W_{\dot{C}\dot{D}\dot{B}a} - \\ & - 64W^2Y^{\dot{A}\dot{B}}W^{\dot{C}\dot{D}}\nabla^{a}_{\dot{A}}W_{\dot{C}\dot{D}\dot{B}a} + \frac{64}{3}iW^2\left(\nabla^{\dot{A}a}Y_{\dot{A}\dot{B}}\right)W^{\dot{B}\dot{C}}\Lambda_{\dot{C}a} + \\ & + \frac{64}{3}iW^2Y_{\dot{A}\dot{B}}\left(\nabla^{\dot{A}a}W^{\dot{B}\dot{C}}\right)\Lambda_{\dot{C}a} + \frac{64}{3}iW^2Y_{\dot{A}\dot{B}}W^{\dot{B}\dot{C}}\nabla^{\dot{A}a}\Lambda_{\dot{C}a}\,. \end{aligned} \tag{3.5}$$

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All the terms in (3.5) can be immediately computed by using the differential relations in appendix B.2, except for the first two, which require a substantial amount of derivatives to compute.

In order to compute $\nabla_{\dot{A}a}\overline{W}^2$, we use the known relation for $\nabla_{\dot{A}a}W_{\dot{C}\dot{D}}$:

$$\nabla_{\dot{A}a}\overline{W}^2 = 2W^{\dot{C}\dot{D}}\nabla_{\dot{A}a}W_{\dot{C}\dot{D}} = 4iW^{\dot{C}\dot{D}}W_{\dot{C}\dot{D}\dot{A}a} - \frac{4}{3}W_{\dot{A}\dot{B}}\Lambda_a^{\dot{B}} \,. \tag{3.6}$$

With this result, we can compute

$$\begin{aligned}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2 = {} & 4i\left(\nabla_{\dot{B}b}W^{\dot{C}\dot{D}}\right)W_{\dot{C}\dot{D}\dot{A}a} + 4iW^{\dot{C}\dot{D}}\nabla_{\dot{B}b}W_{\dot{C}\dot{D}\dot{A}a} - \\ & -\frac{4}{3}\left(\nabla_{\dot{B}b}W_{\dot{A}\dot{C}}\right)\Lambda_a^{\dot{C}} - \frac{4}{3}W_{\dot{A}\dot{C}}\nabla_{\dot{B}b}\Lambda_a^{\dot{C}} \,.\end{aligned} \tag{3.7}$$

The field content implicit in these relations may be seen from the differential relations in appendix B.2.

From (3.7), we can proceed computing:

$$\begin{aligned}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2 = {} & 4i\left(\nabla_{\dot{C}c}\nabla_{\dot{B}b}W^{\dot{D}\dot{E}}\right)W_{\dot{D}\dot{E}\dot{A}a} - 4i\left(\nabla_{\dot{B}b}W^{\dot{D}\dot{E}}\right)\nabla_{\dot{C}c}W_{\dot{D}\dot{E}\dot{A}a} + \\ & + 4i\left(\nabla_{\dot{C}c}W^{\dot{D}\dot{E}}\right)\nabla_{\dot{B}b}W_{\dot{D}\dot{E}\dot{A}a} + 4iW^{\dot{D}\dot{E}}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{D}\dot{E}\dot{A}a} - \\ & - \frac{4}{3}\left(\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{A}\dot{D}}\right)\Lambda_a^{\dot{D}} - \frac{4}{3}\left(\nabla_{\dot{C}c}W_{\dot{A}\dot{C}}\right)\nabla_{\dot{B}b}\Lambda_a^{\dot{C}} - \\ & - \frac{4}{3}\left(\nabla_{\dot{C}c}W_{\dot{A}\dot{D}}\right)\nabla_{\dot{B}b}\Lambda_a^{\dot{D}} - \frac{4}{3}W_{\dot{A}\dot{D}}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_a^{\dot{D}} \,.\end{aligned} \tag{3.8}$$

We got some second spinorial derivatives of superfields, as expected, which we can compute by differentiating some of the relations in appendix B.2. We list the results in appendix C.1.

From (3.8), we finally get

$$\begin{aligned}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2 = {} & -\frac{4}{3}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{A}\dot{E}}\right)\Lambda_a^{\dot{E}} - \frac{4}{3}\left(\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{D}d}\Lambda_a^{\dot{E}} + \\ & + \frac{4}{3}\left(\nabla_{\dot{D}d}\nabla_{\dot{B}b}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{C}c}\Lambda_a^{\dot{E}} - \frac{4}{3}\left(\nabla_{\dot{B}b}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Lambda_a^{\dot{E}} - \\ & - \frac{4}{3}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{B}b}\Lambda_a^{\dot{E}} + \frac{4}{3}\left(\nabla_{\dot{C}c}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{D}d}\nabla_{\dot{B}b}\Lambda_a^{\dot{E}} - \\ & - \frac{4}{3}\left(\nabla_{\dot{D}d}W_{\dot{A}\dot{E}}\right)\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_a^{\dot{E}} - \frac{4}{3}W_{\dot{A}\dot{E}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_a^{\dot{E}} + \\ & + 4i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W^{\dot{E}\dot{F}}\right)W_{\dot{E}\dot{F}\dot{A}a} + \\ & + 4i\left(\nabla_{\dot{C}c}\nabla_{\dot{B}b}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{D}d}W_{\dot{E}\dot{F}\dot{A}a} - \\ & - 4i\left(\nabla_{\dot{D}d}\nabla_{\dot{B}b}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{A}a} + \\ & + 4i\left(\nabla_{\dot{B}b}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{A}a} + \\ & + 4i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a} - \\ & - 4i\left(\nabla_{\dot{C}c}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{D}d}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a} + \\ & + 4i\left(\nabla_{\dot{D}d}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a} + 4iW^{\dot{E}\dot{F}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a} \,.\end{aligned} \tag{3.9}$$

As expected, we get some third spinorial derivatives of superfields, which we can compute by differentiating some of the relations from the previous section and in appendix B.2. We list the results in appendix C.2.

When a spinor derivative acts on a vector derivative of a superfield, first we commute the two derivatives with the help of the torsions and curvatures listed in the appendix B. Then we use the gauge choices (2.3), (2.4) and (2.5) to write

$$\nabla_\mu| = \nabla_\mu \tag{3.10}$$

∇_μ is a Lorentz and SU(2) covariant derivative. For an arbitrary superfield G we have then

$$\nabla_m G| = e_m^{\ \mu} \nabla_\mu\, G| - \frac{1}{2}\psi_m^{Aa}\, \nabla_{Aa} G| - \frac{1}{2}\psi_m^{\dot{A}a}\, \nabla_{\dot{A}a} G\big| \,. \tag{3.11}$$

The equations we have been obtaining allow us to determine the field content of $\phi|$, by replacing the equations with fewer spinorial derivatives (starting from the differential relations in appendix B.2) in the ones with more spinorial derivatives, and by suitable index contraction and derivative commutation.

This same relations (more precisely, their complex conjugates) will also be useful for calculating the higher θ components of ϕ. Other differential relations will be necessary: the series of spinorial derivatives will act on $\phi|$, which after the full component expansion contains much more fields (not only W_{AB}). These fields will also be acted on by four spinorial derivatives, which we have not computed (and, as we will see next section, we don't need to). The computations are straightforward, like the one we did for $\overline{W}^2$. Rather than performing them, we prefer to deduce some of their properties, using the results we have.

4. The field equations for the auxiliary fields

Having seen how to obtain the supersymmetric $\mathcal{R}^4$ lagrangian in components, we are now in a position to analyze the auxiliary field sector. Our main goal is to figure out which auxiliary fields, with this $\mathcal{R}^4$ correction, do not get spacetime derivatives in the action (i.e. have an algebraic field equation and can be eliminated on-shell), and which do get. We start by the $\mathcal{N} = 2$ case, the lagrangian of which we have been determining. Then we summarize the $\mathcal{N} = 1$ case, which we analyzed in previous works, and we compare the two cases.

4.1 The $\mathcal{N} = 2$ case

We start by recalling that, in pure supergravity, both in $\mathcal{N} = 1$ and in $\mathcal{N} = 2$, the auxiliary fields are equal to 0 on-shell [20].

Just by looking at the differential relations in appendix B.2, it is immediate to conclude that, if a $\nabla_{\dot{B}b} W_{\dot{C}\dot{D}\dot{A}a}$ term shows up in the lagrangian, one gets derivatives of $\Phi^{ab}_{A\dot{B}}$ and of U_m. This term shows up already in $\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$ (which shows up, by itself, in higher θ components of $\overline{\phi}$). Dotted spinor derivatives of $\Phi^{ab}_{A\dot{B}}$ and U_m will introduce the physical superfields $W_{\dot{C}\dot{D}}, W_{\dot{C}\dot{D}\dot{A}a}, Y_{\dot{C}\dot{D}Aa}$ and the auxiliary superfields $Y_{\dot{C}\dot{D}}, X_{ab}, \Lambda_{Aa}$ (but not $\Lambda_{\dot{A}a}$). It also introduces a derivative of the superfield $\rho_{\dot{A}a}$. These superfields get then one derivative - $\rho_{\dot{A}a}$ actually gets two - in the term $\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$.

Just by inspection, we expect these auxiliary fields to have derivatives. To actually compute the coefficient of their derivative terms is a hard task - basically it would be equivalent to computing all the terms in the lagrangian, which would require an enormous amount of algebra. Fortunately, it is possible to compute their leading derivative terms using a simple trick.

In the previous section we obtained an expression for $\phi|$ in terms of spinorial derivatives of superfields. To compute higher θ terms, we act on ϕ with undotted spinor derivatives. From (3.5) we see that these derivatives either act on W^2, $W_{\dot{A}\dot{B}\dot{C}c}$, X_{ab}, $\Lambda_{\dot{C}c}$, $Y_{\dot{A}\dot{B}}$ or their spinor derivatives - giving rise to the equations we got in computing $\phi|$, but complex-conjugated -, or they act on four dotted spinorial derivatives of $\overline{W}^2$. Each undotted derivative can be anticommuted with a dotted one (giving rise to a vector derivative and curvature terms), until it finally acts in $\overline{W}^2$ — resulting 0. The torsion and curvature terms we get in this process, after all undotted derivatives are taken, include other superfields like $U_{A\dot{A}}$, $Y_{AB\dot{C}c}$, the spinor derivatives of which we know from computing $\phi|$. Therefore, the higher θ terms of ϕ are either products of terms we already know from $\phi|$, or overall vector derivatives. Since the higher θ terms are multiplied, in the action (3.2), by a corresponding term of the chiral density, we can integrate by parts the derivatives in the higher θ terms and have them acting in the chiral density! Therefore, all the lower θ^{4-n} terms in the chiral density are acted in the action by n vector derivatives. Looking at the chiral density in (2.28), we conclude that X_{ab} and Y_{mn} have terms with at least two derivative in the action, and Λ_{Aa} has terms with at least one derivative in the action. These terms cannot be eliminated by integration by parts.

Furthermore, U_m and Φ_m^{ab} also get derivatives in the action. The reason is the following: the vector derivatives we get come from superspace and, therefore, are (or can be made) Lorentz and U(2) covariant.[3] The physical theory should be only Lorentz invariant; the U(1) and SU(2) connections should be seen respectively as the U_m and Φ_m^{ab} auxiliary fields. Therefore, terms with n vector (Lorentz and U(2) covariant) derivatives give rise to $n-1$ vector Lorentz covariant derivatives of U_m and Φ_m^{ab}.

Derivatives of P and H_m superfields still have not appeared up to now. By our previous arguments, if these derivatives do not appear in $\phi|$, they do not appear at all, since these superfields do not appear in the chiral density except in the highest θ term. Therefore, all that is left to do is check that $\phi|$ has no derivatives of ρ_A^a, P, their hermitean conjugates and H_m.

4.1.1 Derivatives of P and H_m in $\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$ and $\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$

The P and H_m superfields only appear in $\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$ through I, which belongs to the

[3]In this paper, they are just SU(2) covariant, by our choice. The choice of tangent group in superspace is a matter of convenience. We could have kept the U(2) covariance, and the argument in the text would be valid directly, but we broke this covariance to SU(2). We get them extra U_m terms — U_m was the U(1) connection —, which would be reabsorbed in the derivatives, if the U(2) covariance had been kept.

original Weyl multiplet and is given by (2.24), and its derivatives.[4] Recall that I itself has a divergence $\nabla_{B\dot{B}}H^{B\dot{B}}$ and derivatives $\nabla_{B\dot{B}}\rho_{\dot{A}a}$; it shows up already in the expansion of $\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$ in (3.7), through $\nabla_{\dot{B}b}\Lambda^{\dot{B}b}$. These derivatives already show up at this early stage, and will keep showing up as the calculation goes. We anticipate that we will be interested only in derivatives of fields coming from the nonlinear multiplet. Therefore, from now on when we refer to "derivatives of H_m" we are excluding its divergence.

As we keep going and compute $\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$, we see that the derivatives in which we are interested come only from the $\nabla_{\dot{C}c}I$ term; all the other terms in (3.8), which we computed in terms of known expressions in appendix C.1, contribute at most with $\nabla_{B\dot{B}}H^{B\dot{B}}$ terms.

The complete expansion for $\nabla_{\dot{C}c}I$ is given by (C.4) in appendix C.1. For each term in this equation, we compute in appendix D.1 its content in terms of the derivatives of interest. Now, here a surprise happens: after all the contributions have been analyzed, we conclude that they all precisely cancel in such a way that $\nabla_{\dot{C}c}I$ contains no derivatives of P or of H_m (with the exception of $\nabla_{B\dot{B}}H^{B\dot{B}}$, as before). This result is easily obtained from (C.4) and the expressions from appendix D.1. This shows that derivatives of P and H_m do not come up to $\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$.

4.1.2 Derivatives of P and H_m in $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$

An even bigger surprise is that these derivatives also cancel in those terms from $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$ in which we will be interested. The procedure we took to fully check these cancellations was analogous to the previous case: we expanded each term in (3.9) in terms of known expressions in appendix C.2. For those terms which had derivatives of interest, we computed their derivative content in appendix D.2. We summed all the possible contributions for each term in (3.9) and we list them next. During the partial calculations we see lots of independent terms appearing from different sources, but at the end we found several remarkable cancellations of those terms. The most remarkable one happens in $\nabla_{\dot{D}d}\nabla_{\dot{C}c}I$, which is part of the term with $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_{\dot{A}a}$ in (3.9). As it can be seen from the expansion of $\nabla_{\dot{D}d}\nabla_{\dot{C}c}I$ in equation (C.15), in order to get the derivative content of this term we need contributions from every expression in appendix D. It turns out, though, after a lot of algebra, that all the contributions precisely cancel in such a way that $\nabla_{\dot{D}d}\nabla_{\dot{C}c}I$ has absolutely no derivatives of P or H_m besides $\nabla_{B\dot{B}}H^{B\dot{B}}$, exactly like $\nabla_{\dot{C}c}I$ in the previous section. All these results can be checked using the equations in appendices C and D. After performing all the calculations, the final result is then the following: the only nonvanishing contributions to interesting derivative terms in (3.9) come from

$$\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a} = \frac{1}{4}\varepsilon_{\dot{B}\dot{A}}\varepsilon_{\dot{D}\dot{C}}\varepsilon_{dc}X_{ab}\nabla^{A}_{\underline{\dot{E}}}H_{A\underline{\dot{F}}} + \frac{1}{4}\varepsilon_{\dot{B}\dot{A}}\varepsilon_{\dot{D}\dot{C}}\varepsilon_{dc}\Phi^{A}_{\underline{\dot{E}}ab}\nabla_{A\underline{\dot{F}}}P + \cdots . \quad (4.1)$$

[4] We will be only looking for derivatives of P because, although this superfield is chiral, it is generated (and acquires derivatives) through $\nabla_{\dot{B}b}\rho_{\dot{A}a}$ and $\nabla_{\dot{B}b}H_{A\dot{A}}$. Its complex conjugate, $\overline{P}$, does not get derivatives in ϕ, but gets them in $\overline{\phi}$.

These terms actually do not vanish in $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$, but they do vanish in ϕ. This is because, as it may be seen from (3.5), ϕ only contains the combinations $\nabla^{\dot{A}\underline{a}}\nabla^{\underline{b}}_{\dot{A}}\nabla^{\dot{B}}_{a}\nabla_{\dot{B}b}\overline{W}^2$ and $\nabla^{\underline{\dot{A}}a}\nabla^{\underline{\dot{B}}}_{a}\nabla^{b}_{\dot{A}}\nabla_{\dot{B}b}\overline{W}^2$. In the first combination, we need $\nabla^{\dot{D}}_{\underline{d}}\nabla_{\dot{D}\underline{c}}\nabla^{\dot{B}c}W_{\dot{E}\dot{F}\dot{B}}{}^{d}$, while in the second we need $\nabla^{\underline{\dot{D}}c}\nabla^{\underline{\dot{C}}}_{c}\nabla^{b}_{\dot{D}}W_{\dot{E}\dot{F}\dot{C}b}$. Both these combinations have no derivatives of interest, as one sees from (4.1).

The other terms from (3.5) do not have any interesting derivatives, as it may be easily seen from the equations in appendix B.2. We thus have shown that in $\phi|$ there are not any derivatives of P or H_m, apart from the divergence of this last field.

4.1.3 Higher θ terms of ϕ

We have shown that $\phi|$ contains no derivatives of P or the transversal part of H_m. For that purpose, we needed to compute spinor derivatives of all the superfields of "old minimal" $\mathcal{N} = 2$ supergravity. As one can see from the results of appendix D, not all spinor derivatives of superfields originate the vector derivatives we were looking at; only the spinor derivatives of the "dangerous" superfields $\rho^a_{\dot{A}}$, P, H_m and Φ^{ab}_m originate such derivatives.

In order to compute the higher θ terms of ϕ, we act on (3.5) with undotted spinor derivatives and follow the procedure described in the beginning of this section. We will find only terms that we have already computed but, most important, we will only find spinor derivatives of the "safe" superfields W_{AB}, $W_{\dot{A}\dot{B}\dot{C}c}$, X_{ab}, $\Lambda_{\dot{C}c}$, $Y_{\dot{A}\dot{B}}$, $U_{A\dot{A}}$, $Y_{AB\dot{C}c}$, or of their spinor derivatives, but not of any of the "dangerous" ones. These superfields are "safe" even in the sense that they are "closed" with respect to differentiation: spinor derivatives of "safe" superfields only originate "safe" superfields.

This way, we conclude there are no derivatives of P or the transversal part of H_m in the $\mathcal{N} = 2$ supersymmetrization of the fourth power of the Weyl tensor.

4.1.4 The behavior of ρ^a_A

Our results indicate that in $\mathcal{N} = 2$ "old minimal" supergravity with the $\mathcal{R}^4$ correction, the bosonic auxiliary fields from the compensating nonlinear multiplet do not get derivatives and can still be eliminated. Auxiliary fields from the Weyl (Y_{mn}, Λ_{Aa}, U_m, Φ^{ab}_m) and vector (X_{ab}) multiplets get derivatives and cannot be eliminated.

The only unclear result is the behavior of the fundamental (in terms of which all the others are defined) auxiliary field ρ^a_A. Derivatives of ρ^a_A (and of its hermitean conjugate) are not generated by the process of integration by parts we mentioned, but they are constantly being generated in the computation of $\phi|$ already from its beginning. This is because these derivatives exist already in the simple differential relations of appendix B.2 and in the definition (2.24) of I. This is why we have not fully calculated these derivatives, as we did with the other superfields: that would require computing a big number of terms and, for each term, a huge number of different contributions. This is probably because ρ^a_A belongs to a nonlinear multiplet. It would seem a miracle that all these derivative terms would cancel, but we have not shown that they do not cancel and we cannot rule it out! We can at least provide arguments supporting the hypothesis of cancellation. And there are at least two good ones. The first argument is that it seems strange (although it is not impossible) to have a field (ρ^a_A) with a dynamical field equation while having two fields obtained from its spinorial derivatives (P and H_m — see appendix B.1) without such an equation. The second argument is that ρ^a_A, like P and H_m, are intrinsic to the "old minimal" version of $\mathcal{N}=2$ supergravity; they all come, as we saw, from the same multiplet. The physical theory does not depend on these auxiliary fields and, therefore, it seems natural that they can be eliminated from the classical theory and its quantum corrections. We checked that P and (transversal) H_m can be eliminated; the same should be expected for ρ^a_A. If that was the case that the derivatives of ρ^a_A would cancel, its field equation would be some function of the "dynamical" auxiliary fields and their derivatives, such that when replaced in their definitions in appendix B.1, it would result in differential (i.e. dynamical) field equations for these fields.

4.2 The $\mathcal{N}=1$ case

The fields of $\mathcal{N}=1$ conformal supergravity multiplet are the graviton e^m_μ, the gravitino ψ^A_μ and a U(1) gauge field A_μ. (The dilation gauge field B_μ can be gauged away.) Just with these fields, the superconformal algebra closes off-shell. Each one of these is a gauge field; the corresponding gauge invariances must be considered when counting the number of degrees of freedom. In particular, A_μ has 4-1=3 degrees of freedom [21, 22].

To obtain the "old minimal" formulation of $\mathcal{N}=1$ Poincaré supergravity [23, 24], we take the superconformally invariant action of a chiral multiplet. In order to break the superconformal and local U(1) invariances, one must impose some constraint which restricts the parameters of their transformation rules to the chiral multiplet. In superspace this is achieved by imposing the following nonconformal torsion constraint [25]:

$$T_{Am}{}^m = 0\,.$$

This constraint implies the known off-shell constraints and differential relations between the $\mathcal{N}=1$ supergravity superfields R, G_m, W_{ABC}:

$$\nabla^A R = 0 \tag{4.2}$$

$$\nabla^A G_{A\dot{B}} = \frac{1}{24}\nabla_{\dot{B}} R \tag{4.3}$$

$$\nabla^A W_{ABC} = i\left(\nabla_{B\dot{A}} G_C{}^{\dot{A}} + \nabla_{C\dot{A}} G_B{}^{\dot{A}}\right) \tag{4.4}$$

which imply the relation

$$\nabla^2\overline{R} - \overline{\nabla}^2 R = 96 i \nabla^n G_n\,. \tag{4.5}$$

The (anti)chirality condition on $R, \overline{R}$ implies their $\theta=0$ components (resp. the auxiliary fields $M-iN, M+iN$) lie in antichiral/chiral multiplets; (4.3) shows the spin-1/2 parts of the gravitino lie on the same multiplets and, according to (4.5), so does $\partial^\mu A_\mu$ (because $G_m| = A_m$).[5]

[5]The remaining scalar off-shell degree of freedom is the trace of the metric.

In previous works, we have considered a similar problem to the one in the present paper: supersymmetrizing the fourth power of the Weyl tensor, $\mathcal{W}_+^2\mathcal{W}_-^2$, in the "old minimal" formulation of $\mathcal{N}=1$ supergravity [8]. When we took the superspace action which included this term, we obtained algebraic field equations for $R, \overline{R}$. According to (4.5), $\nabla^n G_n$ also obbeys an algebraic equation. The auxiliary fields that belong to the compensating multiplet can thus still be eliminated. This is not the case for the auxiliary fields which come from the Weyl multiplet (A_m), as we obtained, in the same work, a differential field equation for G_m.

4.3 Possible generalizations

It would be interesting to figure out how the results we got can be generalized. Both $\mathcal{N}=1$ and $\mathcal{N}=2$ supergravities admit other minimal formulations, with different choices of compensating multiplets and different sets of auxiliary fields. In "new minimal" $\mathcal{N}=1$ [26], the chiral compensating multiplet is replaced by a compensating tensor multiplet that still breaks conformal invariance but leaves the local U(1) invariance unbroken. In the "new minimal" $\mathcal{N}=2$ [15], one still has the compensating vector multiplet that breaks conformal and local U(1) invariances, but the nonlinear multiplet is replaced by an "improved tensor" compensating multiplet that breaks local SU(2) to local SO(2).

The obvious observation is that both in $\mathcal{N}=1$ and $\mathcal{N}=2$ the auxiliary fields from the tensor multiplets do not get derivatives in the supersymmetric $\mathcal{R}^4$ action, while the auxiliary fields from the Weyl (and vector in $\mathcal{N}=2$) multiplets do get. This way we notice that, in the cases we analyzed, the auxiliary fields that can be eliminated come from multiplets which, on-shell, have no physical fields; while the auxiliary fields that get derivatives come from multiplets with physical fiels on-shell (the graviton, the gravitino(s) and, in $\mathcal{N}=2$, the vector). Our general conjecture for $\mathcal{R}^4$ supergravity, which is fully confirmed in the "old minimal" $\mathcal{N}=1$ case, can now be stated: the auxiliary fields which come from multiplets with on-shell physical fields cannot be eliminated, but the ones that come from compensating multiplets that, on shell, have no physical fields, can.

This analysis should also be extended to nonminimal versions of these theories. These nonminimal versions would have fermionic auxiliary fields (also in $\mathcal{N}=1$). This should be part of another project, and we leave more definite results to another work. Maybe by understanding the behavior of these fermionic auxiliary fields (and the ones in "new minimal" $\mathcal{N}=2$) we could fully understand what in this paper we have just conjectured: the behavior of the auxiliary field ρ_A^a in "old minimal" $\mathcal{N}=2$ supergravity. Possibly the computations would get easier in other versions of the $\mathcal{N}=2$ theory; the fact that they were difficult in the "old minimal" formulation, particularly the ones concerning ρ_A^a, is probably simply due to the presence of the nonlinear multiplet.

A generalization of these results to $\mathcal{N}=3,4$ Poincaré supergravity theories, which can also be seen as broken superconformal theories, is more difficult. This is because these theories do not have an off-shell formulation in conventional superspace. A formulation like this could still be possible, but either in harmonic superspace or with multiplets with central charge.

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5. Conclusions

We wrote down an action containing an $\mathcal{R}^4$ correction to "old minimal" $\mathcal{N}=2$ supergravity. We analyzed its auxiliary field sector, and we concluded that the auxiliary fields belonging to the Weyl and compensating vector multiplets acquire derivatives with these correction and cannot be eliminated on-shell. We checked that all the terms with derivatives for the bosonic auxiliary fields from the compensating nonlinear multiplets cancel; we argued that the same should be true for the fermionic auxiliary field from this multiplet, although we have not performed the full calculation in order to reach a definitive conclusion.

In "old minimal" $\mathcal{N}=1$ supergravity a similar result is valid: the auxiliary field from the Weyl multiplet cannot be eliminated on-shell with the $\mathcal{R}^4$ correction, while the ones from the chiral compensating multiplet can. We then conjectured that analogous results about the Weyl and compensating multiplets should be valid for the other versions of $\mathcal{N}=1,2$ supergravity. In general, we conjecture that auxiliary fields which come from multiplets with on-shell physical fields cannot be eliminated, but those ones that come from compensating multiplets without any on shell physical fields can be eliminated. These results should help to clarify the structure of the supersymmetric $\mathcal{R}^4$ actions in more complicated and less understood theories, either with more supersymmetries (in $d=4$) or in higher dimensions.

The direct supersymmetrization of higher order terms in 10 and 11 dimensions has been an active topic of research, although lots of questions remain open. Some superinvariants associated with the $\mathcal{R}^4$ term have been studied [27]–[33], but complete supersymmetric effective actions including all the leading order corrections to supergravity are still lacking. In M-theory, because of the absence of a microscopical formulation, the construction of superinvariants would be even more important. Hopefully the results we have been getting in four dimensions will provide some insight for the higher dimensional theories!

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A. $\mathcal{N}=2$ SU(2) superspace conventions

We work with standard SU(2) $\mathcal{N}=2$ superspace. We define

$$V_M = \left(V_m, V_{Aa}, V_{\dot{B}b}\right) \tag{A.1}$$

$A, \dot{B}$ are spinor indices, the algebra of which being exactly the same as the $\mathcal{N}=1$ case, which is fully explained in [7, 8]. a is an internal SU(2) index, which is raised and lowered

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with an SU(2)-invariant ε^{ab} tensor, just like the spinor indices: $T^a = \varepsilon^{ab}T_b, T_a = T^b\varepsilon_{ba}$. We take $\varepsilon_{12} = 1$. The basic rule of our conventions (different from other conventions in the literature) is that we use the northwest rule in every index (spinor or SU(2)) contraction. The complex conjugation rules are

$$\overline{V^a_A} = -V_{\dot{A}a}, \overline{V^{Aa}} = V^{\dot{A}}_a, \overline{V_{Aa}} = V^a_{\dot{A}}, \overline{V^A_a} = -V^{\dot{A}a}, \overline{\varepsilon_{ab}} = \varepsilon^{ab}, \overline{\chi^{Aa}\psi^{Bb}} = -\overline{\chi^{Aa}}\ \overline{\psi^{Bb}}\,. \tag{A.2}$$

All other conventions regarding spacetime metrics, the Riemann tensor, Pauli matrices, superspace torsions and curvatures are the same as in [7, 8].

B. Solution to the Bianchi identities in $\mathcal{N} = 2$ SU(2) superspace

In conformal supergravity, all torsions and curvatures can be expressed in terms of the basic superfields W_{AB}, Y_{AB}, $U_{A\dot{A}}$, X_{ab}, their complex conjugates and their covariant derivatives. After breaking of superconformal invariance and local U(2), the basic superfields in the Poincaré theory become the physical field W_{AB} and the auxiliary field ρ^a_A. All torsions and curvatures can be expressed in terms of these superfields, their complex conjugates and their covariant derivatives. In sections B.1 and B.2 we present the definitions of these superfields, and then we list the torsions and curvatures.

B.1 Definitions

$\rho^a_A|$ is an auxiliary field; $W_{AB}|$, at the linearized level, is related to the field strength of the physical vector field A_μ. From (2.17), the complete expression is

$$W_{AB}| = -\frac{i}{2\sqrt{2}}\sigma^{mn}_{AB}F_{mn} - Y_{AB} - \frac{i}{4}\sigma^{mn}_{AB}\left(\psi^{Cc}_m\psi_{nCc} + \psi^{\dot{C}c}_m\psi_{n\dot{C}c}\right). \tag{B.1}$$

Now we present the definitions of the superfields of "old minimal" $\mathcal{N} = 2$ supergravity in terms of W_{AB} and ρ^a_A. The hermitean conjugates can be easily obtained from the basic rules in (A.2), which are valid for ∇^a_A and ρ^a_A, and the definition

$$W_{\dot{A}\dot{B}} = \overline{W_{AB}}\,. \tag{B.2}$$

$$X^{ab} = \frac{1}{2}\left(\nabla^{\dot{A}\underline{a}} - 2\rho^{\dot{A}\underline{a}}\right)\rho^{\underline{b}}_{\dot{A}} = \frac{1}{2}\left(\nabla^{A\underline{a}} - 2\rho^{A\underline{a}}\right)\rho^{\underline{b}}_{A} \tag{B.3}$$

$$Y_{AB} = -\frac{i}{2}\left(\nabla^a_{\underline{A}} + 2\rho^a_{\underline{A}}\right)\rho_{\underline{B}a} \tag{B.4}$$

$$U_{A\dot{A}} = \frac{1}{4}\left(\nabla^a_A\rho_{\dot{A}a} + \nabla^a_{\dot{A}}\rho_{Aa} + 4\rho^a_A\rho_{\dot{A}a}\right) \tag{B.5}$$

$$\Phi^{ab}_{A\dot{A}} = \frac{i}{2}\left(\nabla^{\underline{a}}_A\rho^{\underline{b}}_{\dot{A}} - \nabla^{\underline{a}}_{\dot{A}}\rho^{\underline{b}}_A - 4\rho^{\underline{a}}_A\rho^{\underline{b}}_{\dot{A}}\right) \tag{B.6}$$

$$P = i\nabla^{\dot{A}a}\rho_{\dot{A}a} \tag{B.7}$$

$$H_{A\dot{A}} = -i\nabla^a_A\rho_{\dot{A}a} + i\nabla^a_{\dot{A}}\rho_{Aa} \tag{B.8}$$

$$\Lambda^{Aa} = -i\nabla^A_b X^{ab} \tag{B.9}$$

$$I = i\nabla^{\dot{A}a}\Lambda_{\dot{A}a} - i\nabla^{Aa}\Lambda_{Aa} \tag{B.10}$$

$$W_{BCAa} = \frac{i}{2}\nabla_{Aa}W_{BC} - \frac{i}{6}\left(\varepsilon_{AB}\Lambda_{Ca} + \varepsilon_{AC}\Lambda_{Ba}\right) \tag{B.11}$$

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$$Y_{BC\dot{A}a} = -\frac{i}{2}\nabla_{\dot{A}a}Y_{BC} \tag{B.12}$$

$$W_{ABCD} = \left(\frac{i}{4}\nabla^b_{\underline{A}}\nabla_{\underline{B}b} - 2Y_{\underline{AB}}\right)W_{\underline{CD}} \tag{B.13}$$

$$W_{\dot{A}\dot{B}\dot{C}\dot{D}} = \left(\frac{i}{4}\nabla^b_{\underline{\dot{A}}}\nabla_{\underline{\dot{B}}b} - 2Y_{\underline{\dot{A}\dot{B}}}\right)W_{\underline{\dot{C}\dot{D}}} \tag{B.14}$$

$$\begin{aligned} P_{AB\dot{C}\dot{D}} = {} & \frac{i}{8}\nabla^b_{\underline{A}}\nabla_{\underline{B}b}Y_{\dot{C}\dot{D}} + \frac{i}{8}\nabla^b_{\underline{\dot{C}}}\nabla_{\underline{\dot{D}}b}Y_{AB} - Y_{AB}Y_{\dot{C}\dot{D}} - W_{AB}W_{\dot{C}\dot{D}} - \\ & - U_{\underline{A\dot{C}}}U_{\underline{B\dot{D}}} \end{aligned} \tag{B.15}$$

$$\begin{aligned} R = {} & \frac{i}{4}\left(\nabla^{\dot{A}a}\nabla^{\dot{B}}_a W_{\dot{A}\dot{B}} + \nabla^{Aa}\nabla^B_a W_{AB} + i\nabla^{\dot{A}a}\nabla^b_{\dot{A}}X_{ab} + i\nabla^{Aa}\nabla^b_A X_{ab}\right) - \\ & -2\left(W^{AB}Y_{AB} + W^{\dot{A}\dot{B}}Y_{\dot{A}\dot{B}}\right) - 6X^{ab}X_{ab} + 6U^2 \end{aligned} \tag{B.16}$$

$X^{ab}|, Y_{AB}|, U_m|, \Phi^{ab}_{A\dot{A}}|, P|, H_{A\dot{A}}|$ are auxiliary fields; I is a dependent field. W_{ABCD} is symmetric in all its indices, but W_{ABCc} and $Y_{AB\dot{C}c}$ are only symmetric in A, B. In the linearized approximation, $W_{ABCc}|$ and $Y_{AB\dot{C}c}|$ are the gravitino curls and $W_{ABCD}|$, $P_{AB\dot{C}\dot{D}}|$, $R|$ are the antiself-dual Weyl tensor, the traceless Ricci tensor and the Ricci scalar, respectively:

$$W_{ABCc}| = -\frac{1}{4}\psi_{ABCc} + \cdots \tag{B.17}$$

$$Y_{\dot{A}\dot{B}Cc}| = -\frac{1}{8}\psi_{\dot{A}\dot{B}Cc} + \cdots \tag{B.18}$$

$$W_{ABCD}| = -\frac{1}{8}\mathcal{W}^+_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \cdots \tag{B.19}$$

$$P_{CD\dot{A}\dot{B}}| = \frac{1}{2}\sigma^\mu_{\underline{C\dot{A}}}\sigma^\nu_{\underline{D\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \cdots \tag{B.20}$$

$$R| = -\mathcal{R} + \cdots . \tag{B.21}$$

B.2 Off-shell differential relations

These off-shell differential relations among superfields are direct consequences of the Bianchi identities and the definitions in B.1.

$$\nabla^a_A\rho^b_B = \frac{i}{4}\varepsilon_{AB}\varepsilon^{ab}\overline{P} - \varepsilon_{AB}X^{ab} - i\varepsilon^{ab}Y_{AB} + 2\rho^b_A\rho^a_B \tag{B.22}$$

$$\nabla^a_A\rho^b_{\dot{B}} = -\frac{i}{4}\varepsilon^{ab}H_{A\dot{B}} - \varepsilon^{ab}U_{A\dot{B}} - i\Phi^{ab}_{A\dot{B}} + 2\rho^b_A\rho^a_{\dot{B}} \tag{B.23}$$

$$\nabla_{\dot{C}c}W_{AB} = 0, \nabla_{Cc}W_{\dot{A}\dot{B}} = 0 \tag{B.24}$$

$$\nabla_{Aa}W_{BC} = -2iW_{BCAa} + \frac{1}{3}\left(\varepsilon_{AB}\Lambda_{Ca} + \varepsilon_{AC}\Lambda_{Ba}\right) \tag{B.25}$$

$$\nabla_{Aa}X_{bc} = \frac{i}{3}\left(\varepsilon_{ab}\Lambda_{Ac} + \varepsilon_{ac}\Lambda_{Ab}\right) \tag{B.26}$$

$$\nabla_{Aa}Y_{BC} = -\frac{1}{3}\left(\varepsilon_{AB}\Lambda_{Ca} + \varepsilon_{AC}\Lambda_{Ba}\right) \tag{B.27}$$

$$\nabla_{\dot{A}a}Y_{BC} = 2iY_{BC\dot{A}a} \tag{B.28}$$

$$\nabla_{Aa}U_{B\dot{B}} = Y_{AB\dot{B}a} + \varepsilon_{AB}W^{\dot{A}}{}_{\dot{B}\dot{A}a} + \frac{2}{3}i\varepsilon_{AB}\Lambda_{\dot{B}a} \tag{B.29}$$

$$\nabla^a_{\dot{A}}P = 0, \nabla^a_A\overline{P} = 0 \tag{B.30}$$

$$
\begin{aligned}
\nabla^a_A P \;=\;& -8iW_{AB}{}^{Ba} - \frac{4}{3}\Lambda^a_A - 8\rho^{Ba}W_{AB} + 2P\rho^a_A + 2\rho^{\dot{A}a}H_{A\dot{A}} - \\
& - 4i\rho^{\dot{A}a}U_{A\dot{A}} - 8\nabla_{A\dot{A}}\rho^{\dot{A}a} - 8\Phi^{ab}_{A\dot{A}}\rho^{\dot{A}}_b
\end{aligned} \tag{B.31}
$$

$$
\begin{aligned}
]\nabla^a_A H_{B\dot{B}} \;=\;& 8i\varepsilon_{AB}W_{\dot{A}\dot{B}}{}^{\dot{A}a} + \frac{4}{3}\varepsilon_{AB}\Lambda^a_{\dot{B}} - 4\varepsilon_{AB}\rho^{\dot{A}a}W_{\dot{A}\dot{B}} - 4i\varepsilon_{AB}\rho_{\dot{B}b}X^{ba} - \\
& - 2\varepsilon_{AB}\overline{P}\rho^a_{\dot{B}} - 2\varepsilon_{AB}\rho^{Ca}H_{C\dot{B}} + 4\rho^a_{\dot{B}}Y_{AB} + 2i\rho^a_A U_{B\dot{B}} - \\
& - 8\rho_{Ab}\Phi^{ab}_{B\dot{B}} + 8\rho_{Bb}\Phi^{ab}_{A\dot{B}} - 4\nabla_{B\dot{B}}\rho^a_A + 8\nabla_{A\dot{B}}\rho^a_B
\end{aligned} \tag{B.32}
$$

$$
\begin{aligned}
\nabla^a_A\Phi^{bc}_{B\dot{B}} \;=\;& 2i\varepsilon_{AB}\varepsilon^{a\underline{b}}W_{\dot{B}\dot{A}}{}^{\dot{A}\underline{c}} + \frac{2}{3}\varepsilon_{AB}\varepsilon^{a\underline{b}}\Lambda^{\underline{c}}_{\dot{B}} - 2\varepsilon_{AB}\varepsilon^{a\underline{b}}\rho^{\dot{A}\underline{c}}W_{\dot{A}\dot{B}} + \\
& + 2i\varepsilon^{a\underline{b}}Y_{AB\dot{B}}{}^{\underline{c}} - 2\varepsilon^{a\underline{b}}Y_{AB}\rho^{\underline{c}}_{\dot{B}} + i\varepsilon^{a\underline{b}}\rho^{\underline{c}}_A U_{B\dot{B}} - 2i\varepsilon^{a\underline{b}}\rho^{\underline{c}}_B U_{A\dot{B}} - \\
& - 4\varepsilon^{a\underline{b}}\rho_{Ad}\Phi^{d\underline{c}}_{B\dot{B}} - 2\varepsilon^{a\underline{b}}\nabla_{B\dot{B}}\rho^{\underline{c}}_A - 2i\varepsilon_{AB}X^{a\underline{b}}\rho^{\underline{c}}_{\dot{B}} + 2\rho^{\underline{b}}_A\Phi^{a\underline{c}}_{B\dot{B}}
\end{aligned} \tag{B.33}
$$

$$
\begin{aligned}
\nabla^a_A\Lambda^b_B \;=\;& \frac{i}{8}I\varepsilon_{AB}\varepsilon^{ab} + \frac{3}{2}\varepsilon_{AB}\varepsilon^{ab}\nabla_{C\dot{C}}U^{C\dot{C}} + \\
& + \frac{3}{2}i\varepsilon_{AB}\varepsilon^{ab}\left(Y^{CD}W_{CD} - Y^{\dot{C}\dot{D}}W_{\dot{C}\dot{D}}\right) - 6Y_{AB}X^{ab}
\end{aligned} \tag{B.34}
$$

$$
\nabla^a_A\Lambda^b_{\dot{B}} = 3i\varepsilon^{ab}\left(\nabla_{B\dot{B}}W_A{}^B + \nabla_A{}^{\dot{A}}Y_{\dot{A}\dot{B}}\right) - 3\nabla_{A\dot{B}}X^{ab} - 6iU_{A\dot{B}}X^{ab} \tag{B.35}
$$

$$
\begin{aligned}
\nabla_{\dot{B}b}W_{\dot{C}\dot{D}\dot{A}a} \;=\;& \varepsilon_{ba}W_{\dot{C}\dot{D}\dot{A}\dot{B}} + \frac{1}{12}\varepsilon_{ba}\left(\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}\right)R - \varepsilon_{\dot{B}\dot{A}}\nabla^C{}_{\underline{\dot{C}}}\Phi_{Cb\underline{\dot{D}}a} + \\
& + i\varepsilon_{ba}\varepsilon_{\underline{\dot{C}\dot{A}}}\nabla^C{}_{\underline{\dot{D}}}U_{C\underline{\dot{B}}} + 2\varepsilon_{\dot{B}\dot{A}}W_{\dot{C}\dot{D}\dot{E}\underline{b}}\rho^{\dot{E}}_{\underline{a}} - 2\varepsilon_{\dot{B}\dot{A}}Y_{\dot{C}\dot{D}E\underline{b}}\rho^E_{\underline{a}} + \\
& + \varepsilon_{\dot{B}\dot{A}}\Phi^{Ce}{}_{\underline{\dot{C}b}}\Phi_{Ce\underline{\dot{D}a}} + 2\varepsilon_{ba}Y_{\underline{\dot{C}}\dot{B}}W_{\underline{\dot{D}}\dot{A}} + \varepsilon_{ba}\varepsilon_{\underline{\dot{C}}\dot{B}}\varepsilon_{\underline{\dot{D}}\dot{A}}W^{CD}Y_{CD} + \\
& + \varepsilon_{ba}\varepsilon_{\underline{\dot{C}}\dot{B}}\varepsilon_{\underline{\dot{D}}\dot{A}}X^2 + \varepsilon_{ba}\varepsilon_{\underline{\dot{C}}\dot{B}}\varepsilon_{\underline{\dot{D}}\dot{A}}U^2 + 2i\varepsilon_{\underline{\dot{D}}\dot{A}}Y_{\underline{\dot{C}}\dot{B}}X_{ba} + \\
& + 2i\varepsilon_{\underline{\dot{C}}\dot{B}}W_{\underline{\dot{D}}\dot{A}}X_{ba}
\end{aligned} \tag{B.36}
$$

$$
\begin{aligned}
\nabla_{Bb}W_{\dot{C}\dot{D}\dot{A}a} \;=\;& \varepsilon_{ab}\nabla_{B\dot{A}}W_{\dot{C}\dot{D}} + \varepsilon_{ab}\varepsilon_{\dot{A}\underline{\dot{C}}}\nabla_B{}^{\dot{B}}W_{\dot{B}\underline{\dot{D}}} - \varepsilon_{ab}\varepsilon_{\dot{A}\underline{\dot{C}}}\nabla^C{}_{\underline{\dot{D}}}Y_{BC} \\
& + i\varepsilon_{\dot{A}\underline{\dot{C}}}\nabla_{B\underline{\dot{D}}}X_{ab} - 2\varepsilon_{\dot{A}\underline{\dot{C}}}U_{B\underline{\dot{D}}}X_{ab} + i\varepsilon_{ab}U_{B\dot{A}}W_{\dot{C}\dot{D}} + \\
& + 2i\varepsilon_{ab}\varepsilon_{\dot{A}\underline{\dot{C}}}U_B{}^{\dot{B}}W_{\dot{B}\underline{\dot{D}}}
\end{aligned} \tag{B.37}
$$

$$
\begin{aligned}
\nabla_{\dot{B}b}Y_{CD\dot{A}a} \;=\;& \varepsilon_{ba}P_{CD\dot{B}\dot{A}} + i\varepsilon_{ba}\nabla_{\underline{C\dot{A}}}U_{\underline{D\dot{B}}} - \varepsilon_{\dot{B}\dot{A}}\nabla_{\underline{C}}{}^{\dot{C}}\Phi_{\underline{D}a\dot{C}b} - \\
& - 2\varepsilon_{\dot{B}\dot{A}}W_{CDE\underline{b}}\rho^E_{\underline{a}} + 2\varepsilon_{\dot{B}\dot{A}}Y_{CD\dot{E}\underline{b}}\rho^{\dot{E}}_{\underline{a}} + \varepsilon_{\dot{B}\dot{A}}\Phi_{\underline{Cb}}{}^{\dot{C}e}\Phi_{\underline{Da}\dot{C}e} + \\
& + \varepsilon_{ba}W_{CD}W_{\dot{B}\dot{A}} + \varepsilon_{ba}Y_{CD}Y_{\dot{B}\dot{A}} - \varepsilon_{ba}\varepsilon_{\dot{B}\dot{A}}W_{\underline{C}}{}^EY_{E\underline{D}} - \\
& -i\varepsilon_{\dot{B}\dot{A}}X_{ab}Y_{CD} + i\varepsilon_{\dot{B}\dot{A}}X_{ab}W_{CD} + \varepsilon_{ba}U_{\underline{C\dot{A}}}U_{\underline{D\dot{B}}}
\end{aligned} \tag{B.38}
$$

$$
\begin{aligned}
\nabla_{Bb}Y_{CD\dot{A}a} \;=\;& \varepsilon_{ba}\varepsilon_{B\underline{C}}\nabla_{\underline{D}}{}^{\dot{C}}W_{\dot{C}\dot{A}} - \varepsilon_{ba}\nabla_{\underline{D}\dot{A}}Y_{B\underline{C}} + i\varepsilon_{B\underline{C}}\nabla_{\underline{D}\dot{A}}X_{ab} - \\
& - i\varepsilon_{ba}Y_{CD}U_{B\dot{A}} + 2i\varepsilon_{ba}Y_{B\underline{C}}U_{\underline{D}\dot{A}} + 2\varepsilon_{B\underline{C}}U_{\underline{D}\dot{A}}X_{ba}
\end{aligned} \tag{B.39}
$$

$$
\begin{aligned}
\nabla_{\dot{E}e}W_{\dot{A}\dot{B}\dot{C}\dot{D}} \;=\;& -2i\varepsilon_{\dot{E}\underline{\dot{C}}}\nabla^E{}_{\underline{\dot{C}}}Y_{\underline{\dot{A}\dot{B}}Ee} - 4iY_{\dot{E}\underline{\dot{C}}}W_{\underline{\dot{A}\dot{B}\dot{D}}e} - 8\varepsilon_{\dot{E}\underline{\dot{C}}}X_e{}^fW_{\underline{\dot{A}\dot{B}\dot{D}}f} + \\
& + 3\varepsilon_{\dot{E}\underline{\dot{C}}}Y_{\underline{\dot{A}\dot{B}}Ee}U^E{}_{\underline{\dot{D}}} - 2iW_{\underline{\dot{A}\dot{B}}\dot{E}e}Y_{\underline{\dot{C}\dot{D}}}
\end{aligned} \tag{B.40}
$$

$$
\nabla_{Aa}W_{\dot{A}\dot{B}\dot{C}\dot{D}} = 2i\nabla_{A\underline{\dot{C}}}W_{\underline{\dot{A}\dot{B}\dot{D}}a} + U_{A\underline{\dot{C}}}W_{\underline{\dot{A}\dot{B}\dot{D}}a} + 4iW_{\underline{\dot{C}\dot{D}}}Y_{\underline{\dot{A}\dot{B}}Aa} \tag{B.41}
$$

$$
\nabla_{\dot{E}e}P_{CD\dot{B}\dot{A}} = 2i\nabla_{\underline{C}\dot{E}}Y_{\dot{A}\dot{B}\underline{D}e} - i\nabla_{\underline{C\dot{A}}}Y_{\dot{E}\underline{\dot{B}D}e} + i\varepsilon_{\dot{E}\underline{\dot{B}}}\nabla_{\underline{C\dot{A}}}W_{\underline{D}E}{}^E{}_e - 2iY_{CD\dot{E}e}Y_{\dot{A}\dot{B}} -
$$

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$$
\begin{aligned}
&- iY_{\dot{E}\underline{\dot{A}}}Y_{CD\underline{\dot{B}}e} + i\varepsilon_{\dot{E}\underline{\dot{A}}}W_{\underline{C}}{}^{E}Y_{E\underline{D}\dot{\underline{B}}e} - 2iW_{CD}W_{\dot{A}\dot{B}\dot{E}e} - \\
&- i\varepsilon_{\dot{E}\underline{\dot{A}}}W_{CD}W_{\underline{\dot{B}}\dot{C}}{}^{\dot{C}}{}_{e} - 5\varepsilon_{\dot{E}\underline{\dot{A}}}X_{e}{}^{f}Y_{CD\underline{\dot{B}}f} - 2\varepsilon_{\dot{E}\underline{\dot{A}}}W_{CDFe}U^{F}{}_{\underline{\dot{B}}} + \\
&+ \varepsilon_{\dot{E}\underline{\dot{A}}}U_{\underline{C\dot{B}}}W_{\underline{D}E}{}^{E}{}_{e} + \frac{1}{2}U_{\underline{C\dot{B}}}Y_{\dot{E}\underline{\dot{A}D}e} + 2U_{\underline{C}\dot{E}}Y_{\dot{A}\dot{B}\underline{D}e} \qquad \text{(B.42)}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\dot{A}a}R = &-i\nabla^{B\dot{B}}Y_{\dot{A}\dot{B}Ba} + 3i\nabla^{B}{}_{\dot{A}}W_{BC}{}^{C}{}_{a} + 10X_{ab}W_{\dot{A}\dot{B}}{}^{\dot{B}b} - 2iW_{\dot{C}\dot{B}}{}^{\dot{B}}{}_{a}Y_{\dot{A}}{}^{\dot{C}} - \\
&- 4iW_{\dot{C}\dot{B}\dot{A}a}Y^{\dot{C}\dot{B}} - 3iW^{CB}Y_{CB\dot{A}a} + \frac{9}{2}U^{B\dot{B}}Y_{\dot{A}\dot{B}Ba} + \frac{9}{2}U^{B}{}_{\dot{A}}W_{BC}{}^{C}{}_{a}\,. \qquad \text{(B.43)}
\end{aligned}
$$

Using (B.38) and (B.39), one may compute $\nabla^{B\dot{B}}Y_{\dot{A}\dot{B}Ba}$; replacing in (B.43), we get the more convenient expression

$$
\begin{aligned}
\nabla_{\dot{A}a}R = &\; 4i\nabla^{B}{}_{\dot{A}}W_{BC}{}^{C}{}_{a} + 12X_{ab}W_{\dot{A}\dot{B}}{}^{\dot{B}b} - 2iW^{CB}Y_{CB\dot{A}a} - \\
&- 6iW_{\dot{C}\dot{B}\dot{A}a}Y^{\dot{C}\dot{B}} + 12U^{B\dot{B}}Y_{\dot{A}\dot{B}Ba} + 4U^{B}{}_{\dot{A}}W_{BC}{}^{C}{}_{a}\,. \qquad \text{(B.44)}
\end{aligned}
$$

B.3 Torsions

$$
\begin{aligned}
T_{A\dot{B}}^{abm} &= -2i\varepsilon^{ab}\sigma^{m}_{A\dot{B}} \\
T_{AB}^{abm}, T_{\dot{A}\dot{B}}^{abm} &= 0 \\
T_{\dot{A}aBbCc}, T_{Aa\dot{B}b\dot{C}c} &= 0 \\
T_{AaBbCc}, T_{\dot{A}a\dot{B}b\dot{C}c} &= 0 \\
T_{A}^{amn}, T_{\dot{A}}^{amn} &= 0 \\
T_{A\dot{A}BbCc} &= -\frac{i}{2}\varepsilon_{bc}\left(\varepsilon_{AB}U_{C\dot{A}} + \varepsilon_{AC}U_{B\dot{A}}\right) \\
T_{A\dot{A}\dot{B}b\dot{C}c} &= \frac{i}{2}\varepsilon_{bc}\left(\varepsilon_{\dot{A}\dot{B}}U_{A\dot{C}} + \varepsilon_{\dot{A}\dot{C}}U_{A\dot{B}}\right) \\
T_{A\dot{A}Bb\dot{C}c} &= -\varepsilon_{bc}\left(\varepsilon_{AB}W_{\dot{A}\dot{C}} + \varepsilon_{\dot{A}\dot{C}}Y_{AB}\right) - i\varepsilon_{AB}\varepsilon_{\dot{A}\dot{C}}X_{bc} \\
T_{A\dot{A}\dot{B}bCc} &= \varepsilon_{bc}\left(\varepsilon_{AC}Y_{\dot{A}\dot{B}} + \varepsilon_{\dot{A}\dot{B}}W_{AC}\right) + i\varepsilon_{AC}\varepsilon_{\dot{A}\dot{B}}X_{bc} \\
T_{mnp} &= 0 \\
T_{A\dot{A}B\dot{B}Cc} &= -\varepsilon_{\dot{A}\dot{B}}W_{ABCc} - \varepsilon_{AB}Y_{\dot{A}\dot{B}Cc} \\
T_{A\dot{A}B\dot{B}\dot{C}c} &= \varepsilon_{AB}W_{\dot{A}\dot{B}\dot{C}c} + \varepsilon_{\dot{A}\dot{B}}Y_{AB\dot{C}c}\,. \qquad \text{(B.45)}
\end{aligned}
$$

B.4 Lorentz curvatures

$$
\begin{aligned}
R_{AaBbCD} &= -2i\varepsilon_{AB}\varepsilon_{ab}Y_{CD} + 2\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}\right)X_{ab} \\
R_{AaBb\dot{C}\dot{D}} &= -2i\varepsilon_{AB}\varepsilon_{ab}W_{\dot{C}\dot{D}} \\
R_{\dot{A}a\dot{B}bCD} &= -2i\varepsilon_{\dot{A}\dot{B}}\varepsilon_{ab}W_{CD} \\
R_{\dot{A}a\dot{B}b\dot{C}\dot{D}} &= -2i\varepsilon_{\dot{A}\dot{B}}\varepsilon_{ab}Y_{\dot{C}\dot{D}} + 2\left(\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}\right)X_{ab} \\
R_{\dot{A}aBbCD} &= \varepsilon_{ab}\left(\varepsilon_{BC}U_{D\dot{A}} + \varepsilon_{BD}U_{C\dot{A}}\right) \\
R_{\dot{A}aBb\dot{C}\dot{D}} &= -\varepsilon_{ab}\left(\varepsilon_{\dot{A}\dot{C}}U_{B\dot{D}} + \varepsilon_{\dot{A}\dot{D}}U_{B\dot{C}}\right) \\
R_{A\dot{A}BbCD} &= -i\varepsilon_{B\underline{C}}Y_{A\underline{D}\dot{A}b} + i\varepsilon_{AB}Y_{CD\dot{A}b} + \frac{i}{2}\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}\right)W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{b}
\end{aligned}
$$

$$
\begin{aligned}
R_{A\dot{A}\dot{B}bCD} &= i\varepsilon_{A\underline{C}}Y_{\dot{A}\dot{B}\underline{D}b} + 2i\varepsilon_{\dot{A}\dot{B}}W_{\underline{ACD}b} + \frac{i}{3}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{A\underline{C}}W_{\underline{D}}{}^{B}{}_{Bb} \\
R_{A\dot{A}Bb\dot{C}\dot{D}} &= i\varepsilon_{\dot{A}\underline{\dot{C}}}Y_{AB\underline{\dot{D}}b} + 2i\varepsilon_{AB}W_{\underline{\dot{A}\dot{C}\dot{D}}b} + \frac{i}{3}\varepsilon_{AB}\varepsilon_{\dot{A}\underline{\dot{C}}}W_{\underline{\dot{D}}}{}^{\dot{B}}{}_{\dot{B}b} \\
R_{A\dot{A}\dot{B}b\dot{C}\dot{D}} &= -i\varepsilon_{\dot{B}\underline{\dot{C}}}Y_{\dot{A}\underline{\dot{D}}Ab} + i\varepsilon_{\dot{A}\dot{B}}Y_{\dot{C}\dot{D}Ab} + \frac{i}{2}\left(\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}\right)W_{AB}{}^{B}{}_{b} \\
R_{A\dot{A}B\dot{B}CD} &= \varepsilon_{AB}P_{CD\dot{A}\dot{B}} + \frac{1}{12}\varepsilon_{\dot{A}\dot{B}}\left(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}\right)R + \varepsilon_{\dot{A}\dot{B}}W_{ABCD} \\
R_{A\dot{A}B\dot{B}\dot{C}\dot{D}} &= \varepsilon_{\dot{A}\dot{B}}P_{AB\dot{C}\dot{D}} + \frac{1}{12}\varepsilon_{AB}\left(\varepsilon_{\dot{A}\dot{C}}\varepsilon_{\dot{B}\dot{D}} + \varepsilon_{\dot{A}\dot{D}}\varepsilon_{\dot{B}\dot{C}}\right)R + \varepsilon_{AB}W_{\dot{A}\dot{B}\dot{C}\dot{D}}\,.
\end{aligned} \tag{B.46}
$$

B.5 SU(2) curvatures

$$
\begin{aligned}
R_{AaBbcd} &= -2\varepsilon_{AB}\left(\varepsilon_{bd}X_{ac} - \varepsilon_{ad}X_{bc}\right) - 2i\left(\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{ad}\varepsilon_{bc}\right)Y_{AB} \\
R_{\dot{A}a\dot{B}bcd} &= -2\varepsilon_{\dot{A}\dot{B}}\left(\varepsilon_{bd}X_{ac} - \varepsilon_{ad}X_{bc}\right) - 2i\left(\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{ad}\varepsilon_{bc}\right)Y_{\dot{A}\dot{B}} \\
R_{Aa\dot{B}bcd} &= -2\left(\varepsilon_{ac}\varepsilon_{bd} + \varepsilon_{ad}\varepsilon_{bc}\right)U_{A\dot{B}} \\
R_{A\dot{A}Bbcd} &= 2i\varepsilon_{b\underline{d}}Y_{AB\dot{A}\underline{c}} - 2i\varepsilon_{AB}\varepsilon_{b\underline{d}}W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{\underline{c}} - \frac{2}{3}\varepsilon_{AB}\varepsilon_{b\underline{d}}\Lambda_{\dot{A}\underline{c}} \\
R_{A\dot{A}\dot{B}bcd} &= 2i\varepsilon_{b\underline{d}}Y_{\dot{A}\dot{B}A\underline{c}} - 2i\varepsilon_{\dot{A}\dot{B}}\varepsilon_{b\underline{d}}W_{AB}{}^{B}{}_{\underline{c}} - \frac{2}{3}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{b\underline{d}}\Lambda_{A\underline{c}} \\
R_{A\dot{A}B\dot{B}cd} &= -2\varepsilon_{\dot{A}\dot{B}}\left(W_{ABE\underline{c}}\rho^{E}_{\underline{d}} - Y_{AB\dot{E}\underline{c}}\rho^{\dot{E}}_{\underline{d}} + \frac{1}{2}\nabla_{\underline{A}}{}^{\dot{E}}\Phi_{\underline{B}\dot{E}cd} - \frac{1}{2}\Phi_{\underline{A}}{}^{\dot{E}e}{}_{c}\Phi_{\underline{B}\dot{E}\underline{de}}\right) + \\
&\quad + 2\varepsilon_{AB}\left(W_{\dot{A}\dot{B}\dot{E}\underline{c}}\rho^{\dot{E}}_{\underline{d}} - Y_{\dot{A}\dot{B}E\underline{c}}\rho^{E}_{\underline{d}} - \frac{1}{2}\nabla^{E}{}_{\underline{\dot{A}}}\Phi_{E\underline{\dot{B}}cd} + \frac{1}{2}\Phi^{E}{}_{\underline{\dot{A}ce}}\Phi_{E\underline{\dot{B}d}}{}^{e}\right).
\end{aligned} \tag{B.47}
$$

We see that all torsions and curvatures can be expressed in terms of the basic superfields W_{AB}, Y_{AB}, $U_{A\dot{A}}$, X_{ab}, their complex conjugates and their covariant derivatives (see section B.1). This is obvious except for $R_{A\dot{A}B\dot{B}cd}$, which may be rewritten as

$$
\begin{aligned}
R_{A\dot{A}B\dot{B}cd} = \varepsilon_{\dot{A}\dot{B}}&\left(\frac{1}{2}\nabla^{\dot{C}}{}_{\underline{c}}Y_{AB\dot{C}\underline{d}} + iX_{cd}Y_{AB} - iX_{cd}W_{AB}\right) + \\
+\varepsilon_{AB}&\left(\frac{1}{2}\nabla^{C}{}_{\underline{c}}Y_{\dot{A}\dot{B}C\underline{d}} + iX_{cd}Y_{\dot{A}\dot{B}} - iX_{cd}W_{\dot{A}\dot{B}}\right).
\end{aligned} \tag{B.48}
$$

C. The lagrangian in components

C.1 Calculation of $\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$

In this section, we fully express the terms with two spinorial derivatives arising in the calculation of (3.8) as functions of those with one spinorial derivative, previously computed in appendix B. These terms are:

$$
\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{A}\dot{D}} = 2i\nabla_{\dot{C}c}W_{\dot{A}\dot{D}\dot{B}b} - \frac{2}{3}\varepsilon_{\dot{B}\underline{\dot{A}}}\nabla_{\dot{C}c}\Lambda_{\underline{\dot{D}}b} \tag{C.1}
$$

$$
\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{D}\dot{E}\dot{A}a} = -\varepsilon_{ab}\nabla_{\dot{C}c}W_{\dot{D}\dot{E}\dot{A}\dot{B}} - 2\varepsilon_{ab}\left(\nabla_{\dot{C}c}Y_{\dot{B}\underline{\dot{D}}}\right)W_{\underline{\dot{E}}\dot{A}} - 2\varepsilon_{ab}Y_{\dot{B}\underline{\dot{D}}}\nabla_{\dot{C}c}W_{\underline{\dot{E}}\dot{A}} +
$$

$$
\begin{aligned}
&+2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{C}c}W_{\dot{D}\dot{E}\dot{F}\underline{a}}\right)\rho^{\dot{F}}_{\underline{b}}-2\varepsilon_{\dot{B}\dot{A}}W_{\dot{D}\dot{E}\dot{F}\underline{a}}\nabla_{\dot{C}c}\rho^{\dot{F}}_{\underline{b}}-\\
&-2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{C}c}Y_{\dot{D}\dot{E}E\underline{a}}\right)\rho^{E}_{\underline{b}}+2\varepsilon_{\dot{B}\dot{A}}Y_{\dot{D}\dot{E}E\underline{a}}\nabla_{\dot{C}c}\rho^{E}_{\underline{b}}-\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{C}c}\nabla^{E}{}_{\underline{\dot{D}}}\Phi_{E\underline{\dot{E}}ab}+\\
&+2\varepsilon_{\dot{B}\dot{A}}\Phi^{E}{}_{\underline{\dot{D}}}{}^{e}{}_{\underline{a}}\nabla_{\dot{C}c}\Phi_{E\underline{\dot{E}}e\underline{b}}-2i\varepsilon_{\dot{B}\underline{\dot{D}}}\left(\nabla_{\dot{C}c}X_{ab}\right)W_{\underline{\dot{E}}\dot{A}}-\\
&-2i\varepsilon_{\dot{B}\underline{\dot{D}}}X_{ab}\nabla_{\dot{C}c}W_{\underline{\dot{E}}\dot{A}}+\frac{1}{6}\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{D}}}\nabla_{\dot{C}c}R-\\
&-i\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\underline{\underline{\dot{A}}}}\nabla_{\dot{C}c}\nabla^{E}{}_{\underline{\dot{D}}}U_{E\underline{\underline{\dot{B}}}}+\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{D}}}U^{F\dot{F}}\nabla_{\dot{C}c}U_{F\dot{F}}+\\
&+2i\varepsilon_{\underline{\dot{E}}\dot{A}}\left(\nabla_{\dot{C}c}Y_{\dot{B}\underline{\dot{D}}}\right)X_{ab}+2i\varepsilon_{\underline{\dot{E}}\dot{A}}Y_{\dot{B}\underline{\dot{D}}}\nabla_{\dot{C}c}X_{ab}+\\
&+\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{D}}}W^{AB}\nabla_{\dot{C}c}Y_{AB}+2\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{D}}}X^{de}\nabla_{\dot{C}c}X_{de}
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_{\dot{A}a}=\;&\frac{i}{8}\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{C}c}I-\frac{3}{2}\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{C}c}\nabla^{F\dot{F}}U_{F\dot{F}}-\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}W^{AB}\nabla_{\dot{C}c}Y_{AB}+\\
&+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}W^{\dot{E}\dot{F}}\nabla_{\dot{C}c}Y_{\dot{E}\dot{F}}+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}Y^{\dot{E}\dot{F}}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}}+6X_{ab}\nabla_{\dot{C}c}Y_{\dot{B}\dot{A}}+\\
&+6Y_{\dot{B}\dot{A}}\nabla_{\dot{C}c}X_{ab}\,.
\end{aligned}
\tag{C.3}
$$

For (C.3) we need $\nabla_{\dot{C}c}I$, which we compute here.

$$
\begin{aligned}
\nabla_{\dot{C}c}I=\;&4\nabla_{\dot{C}c}R-48X^{ab}\nabla_{\dot{C}c}X_{ab}-12W^{AB}\nabla_{\dot{C}c}Y_{AB}-12W^{\dot{A}\dot{B}}\nabla_{\dot{C}c}Y_{\dot{A}\dot{B}}-\\
&-12Y^{\dot{A}\dot{B}}\nabla_{\dot{C}c}W_{\dot{A}\dot{B}}-24U^{F\dot{F}}\nabla_{\dot{C}c}U_{F\dot{F}}+3P\nabla_{\dot{C}c}\overline{P}+3H^{F\dot{F}}\nabla_{\dot{C}c}H_{F\dot{F}}-\\
&-24\Phi^{F\dot{F}}_{ab}\nabla_{\dot{C}c}\Phi^{ab}_{F\dot{F}}-6\nabla_{\dot{C}c}\nabla^{F\dot{F}}H_{F\dot{F}}-16i\left(\nabla_{\dot{C}c}\rho^{Aa}\right)\Lambda_{Aa}+16i\rho^{Aa}\nabla_{\dot{C}c}\Lambda_{Aa}+\\
&+16i\left(\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)\Lambda_{\dot{A}a}-16i\rho^{\dot{A}a}\nabla_{\dot{C}c}\Lambda_{\dot{A}a}+48\left(\nabla_{\dot{C}c}\rho^{Aa}\right)W_{AB}{}^{B}{}_{a}-\\
&-48\rho^{Aa}\nabla_{\dot{C}c}W_{AB}{}^{B}{}_{a}-48\left(\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}+48\rho^{\dot{A}a}\nabla_{\dot{C}c}W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}-\\
&-96iW^{AB}\rho_{Aa}\nabla_{\dot{C}c}\rho^{a}_{B}-96iW^{\dot{A}\dot{B}}\rho_{\dot{A}a}\nabla_{\dot{C}c}\rho^{a}_{\dot{B}}+48i\rho_{\dot{A}a}\rho^{a}_{\dot{B}}\nabla_{\dot{C}c}W^{\dot{A}\dot{B}}-\\
&-48U^{A\dot{A}}\left(\nabla_{\dot{C}c}\rho_{Aa}\right)\rho^{a}_{\dot{A}}+48U^{A\dot{A}}\rho_{Aa}\nabla_{\dot{C}c}\rho^{a}_{\dot{A}}-48\rho_{Aa}\rho^{a}_{\dot{A}}\nabla_{\dot{C}c}U^{A\dot{A}}+\\
&+48i\left(\nabla_{\dot{C}c}\rho_{Aa}\right)\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}-48i\rho_{Aa}\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}-48i\left(\nabla_{\dot{C}c}\rho_{\dot{A}a}\right)\nabla^{A\dot{A}}\rho^{a}_{A}+\\
&+48i\rho_{\dot{A}a}\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{A}+96i\Phi^{ab}_{A\dot{A}}\rho^{\dot{A}}_{b}\nabla_{\dot{C}c}\rho^{A}_{a}+96i\Phi^{ab}_{A\dot{A}}\rho^{A}_{b}\nabla_{\dot{C}c}\rho^{\dot{A}}_{a}+\\
&+96i\rho^{A}_{a}\rho^{\dot{A}}_{b}\nabla_{\dot{C}c}\Phi^{ab}_{A\dot{A}}\,.
\end{aligned}
\tag{C.4}
$$

Terms involving combinations of vector and spinor covariant derivatives may be written as vector derivatives of the relations listed in appendix B using the commutation relations.

C.2 Calculation of $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\nabla_{\dot{A}a}\overline{W}^2$

In this section, we fully express the terms with two and three spinorial derivatives arising in the calculation of (3.9) as functions of those with one spinorial derivative, previously computed in appendix B. These terms are:

$$
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}}=2i\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{B}b}-\frac{2}{3}\varepsilon_{\dot{B}\underline{\dot{E}}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Lambda_{\underline{\dot{F}}b}
\tag{C.5}
$$

$$
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}W_{\dot{E}\dot{F}\dot{A}a}=-\varepsilon_{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{A}\dot{B}}-2\varepsilon_{ab}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{B}\underline{\dot{E}}}\right)W_{\underline{\dot{F}}\dot{A}}+
$$

$$
\begin{aligned}
&+2\varepsilon_{ab}\left(\nabla_{\dot{C}c}Y_{\dot{B}\underline{\dot{E}}}\right)\nabla_{\dot{D}d}W_{\underline{\dot{F}}\dot{A}}-2\varepsilon_{ab}\left(\nabla_{\dot{D}d}Y_{\dot{B}\underline{\dot{E}}}\right)\nabla_{\dot{C}c}W_{\underline{\dot{F}}\dot{A}}-\\
&-2\varepsilon_{ab}Y_{\dot{B}\underline{\dot{E}}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\underline{\dot{F}}\dot{A}}+2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{G}\underline{a}}\right)\rho^{\dot{G}}_{\underline{b}}+\\
&+2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{G}\underline{a}}\right)\nabla_{\dot{D}d}\rho^{\dot{G}}_{\underline{b}}-2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}W_{\dot{E}\dot{F}\dot{G}\underline{a}}\right)\nabla_{\dot{C}c}\rho^{\dot{G}}_{\underline{b}}+\\
&+2\varepsilon_{\dot{B}\dot{A}}W_{\dot{E}\dot{F}\dot{G}\underline{a}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{\dot{G}}_{\underline{b}}-2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{E}\dot{F}E\underline{a}}\right)\rho^{E}_{\underline{b}}-\\
&-2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{C}c}Y_{\dot{E}\dot{F}E\underline{a}}\right)\nabla_{\dot{D}d}\rho^{E}_{\underline{b}}+2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}Y_{\dot{E}\dot{F}E\underline{a}}\right)\nabla_{\dot{C}c}\rho^{E}_{\underline{b}}-\\
&-2\varepsilon_{\dot{B}\dot{A}}Y_{\dot{E}\dot{F}E\underline{a}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{E}_{\underline{b}}-\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{E}_{\underline{\dot{E}}}\Phi_{E\underline{\dot{F}}ab}+\\
&+2\varepsilon_{\dot{B}\dot{A}}\Phi^{E}{}_{\underline{\dot{E}}}{}^{e}{}_{\underline{a}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Phi_{E\underline{\dot{F}}e\underline{b}}-2\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{C}c}\Phi^{E}{}_{\underline{\dot{E}}}{}^{e}{}_{\underline{a}}\right)\nabla_{\dot{D}d}\Phi_{E\underline{\dot{F}}e\underline{b}}-\\
&-i\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\underline{\underline{\dot{A}}}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{E}_{\underline{\dot{F}}}U_{E\underline{\underline{\dot{B}}}}+\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}\left(\nabla_{\dot{D}d}U^{G\dot{G}}\right)\nabla_{\dot{C}c}U_{G\dot{G}}+\\
&+\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}U^{G\dot{G}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}U_{G\dot{G}}-2i\varepsilon_{\dot{B}\underline{\dot{E}}}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}X_{ab}\right)W_{\underline{\dot{F}}\dot{A}}+\\
&+2i\varepsilon_{\dot{B}\underline{\dot{E}}}\left(\nabla_{\dot{C}c}X_{ab}\right)\nabla_{\dot{D}d}W_{\underline{\dot{F}}\dot{A}}-2i\varepsilon_{\dot{B}\underline{\dot{E}}}\left(\nabla_{\dot{D}d}X_{ab}\right)\nabla_{\dot{C}c}W_{\underline{\dot{F}}\dot{A}}-\\
&-2i\varepsilon_{\dot{B}\underline{\dot{E}}}X_{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\underline{\dot{F}}\dot{A}}-2i\varepsilon_{\dot{A}\underline{\dot{E}}}\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}X_{ab}\right)Y_{\underline{\dot{F}}\dot{B}}+\\
&+2i\varepsilon_{\dot{A}\underline{\dot{E}}}\left(\nabla_{\dot{C}c}X_{ab}\right)\nabla_{\dot{D}d}Y_{\underline{\dot{F}}\dot{B}}-2i\varepsilon_{\dot{A}\underline{\dot{E}}}\left(\nabla_{\dot{D}d}X_{ab}\right)\nabla_{\dot{C}c}Y_{\underline{\dot{F}}\dot{B}}-\\
&-2i\varepsilon_{\dot{A}\underline{\dot{E}}}X_{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\underline{\dot{F}}\dot{B}}+\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}W^{AB}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{AB}+\\
&+2\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}\left(\nabla_{\dot{D}d}X^{ef}\right)\nabla_{\dot{C}c}X_{ef}+\\
&+2\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}X^{ef}\nabla_{\dot{D}d}\nabla_{\dot{C}c}X_{ef}+\\
&+\frac{1}{6}\varepsilon_{ab}\varepsilon_{\underline{\dot{E}}\dot{A}}\varepsilon_{\dot{B}\underline{\dot{F}}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}R
\end{aligned}
\tag{C.6}
$$

$$
\begin{aligned}
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_{\dot{A}a}=&-\frac{3}{2}\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{F\dot{F}}U_{F\dot{F}}-\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}W^{AB}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{AB}+\\
&+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}W^{\dot{E}\dot{F}}\right)\nabla_{\dot{C}c}Y_{\dot{E}\dot{F}}+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}W^{\dot{E}\dot{F}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{E}\dot{F}}+\\
&+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\left(\nabla_{\dot{D}d}Y^{\dot{E}\dot{F}}\right)\nabla_{\dot{C}c}W_{\dot{E}\dot{F}}+\frac{3}{2}i\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}Y^{\dot{E}\dot{F}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}}+\\
&+6\left(\nabla_{\dot{D}d}X_{ab}\right)\nabla_{\dot{C}c}Y_{\dot{B}\dot{A}}+6X_{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{B}\dot{A}}+6Y_{\dot{B}\dot{A}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}X_{ab}+\\
&+6\left(\nabla_{\dot{D}d}Y_{\dot{B}\dot{A}}\right)\nabla_{\dot{C}c}X_{ab}+\frac{i}{8}\varepsilon_{ab}\varepsilon_{\dot{B}\dot{A}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}I\,.
\end{aligned}
\tag{C.7}
$$

These results require knowing second spinorial derivatives of superfields, some of which we have computed in section C.1, but others we have not computed yet. We present those here:

$$
\begin{aligned}
\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{E}\dot{F}\dot{A}\dot{B}}=&-4i\left(\nabla_{\dot{D}d}Y_{\dot{B}\underline{\dot{A}}}\right)W_{\underline{\dot{B}\dot{E}\dot{F}}c}-4iY_{\dot{B}\underline{\dot{A}}}\nabla_{\dot{D}d}W_{\underline{\dot{B}\dot{E}\dot{F}}c}-\\
&-8\varepsilon_{\dot{C}\underline{\dot{F}}}\left(\nabla_{\dot{D}d}X_{c}{}^{e}\right)W_{\underline{\dot{A}\dot{B}\dot{E}}e}-8\varepsilon_{\dot{C}\underline{\dot{F}}}X_{c}{}^{e}\nabla_{\dot{D}d}W_{\underline{\dot{A}\dot{B}\dot{E}}e}-\\
&-2i\left(\nabla_{\dot{D}d}Y_{\underline{\dot{A}\dot{B}}}\right)W_{\underline{\dot{E}\dot{F}}\dot{C}c}-2iY_{\underline{\dot{A}\dot{B}}}\nabla_{\dot{D}d}W_{\underline{\dot{E}\dot{F}}\dot{C}c}-2i\varepsilon_{\dot{C}\underline{\dot{F}}}\nabla_{\dot{D}d}\nabla^{E}_{\underline{\dot{E}}}Y_{\underline{\dot{A}\dot{B}}Ec}+\\
&+3\varepsilon_{\dot{C}\underline{\dot{F}}}\left(\nabla_{\dot{D}d}U^{E}_{\underline{\dot{E}}}\right)Y_{\underline{\dot{A}\dot{B}}Ec}+3\varepsilon_{\dot{C}\underline{\dot{F}}}U^{E}_{\underline{\dot{E}}}\nabla_{\dot{D}d}Y_{\underline{\dot{A}\dot{B}}Ec}
\end{aligned}
\tag{C.8}
$$

$$
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho_{\dot{B}b}=-\varepsilon_{\dot{C}\dot{B}}\nabla_{\dot{D}d}X_{cb}-i\varepsilon_{cb}\nabla_{\dot{D}d}Y_{\dot{B}\dot{C}}+2\rho_{\dot{B}c}\nabla_{\dot{D}d}\rho_{\dot{C}b}-2\rho_{\dot{C}b}\nabla_{\dot{D}d}\rho_{\dot{B}c}
\tag{C.9}
$$

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$$
\begin{aligned}
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho_{Bb} = {} & \frac{i}{4}\varepsilon_{cb}\nabla_{\dot{D}d}H_{B\dot{C}} - \varepsilon_{cb}\nabla_{\dot{D}d}U_{B\dot{C}} + i\nabla_{\dot{D}d}\Phi_{B\dot{C}cb} + 2\rho_{Bc}\nabla_{\dot{D}d}\rho_{\dot{C}b} - \\
& - 2\rho_{\dot{C}b}\nabla_{\dot{D}d}\rho_{Bc} \qquad\qquad (C.10) \\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{A}\dot{B}Aa} = {} & \varepsilon_{ac}\nabla_{\dot{D}d}\nabla_{A\underline{\dot{A}}}Y_{\dot{C}\underline{\dot{B}}} + 2i\varepsilon_{ac}\left(\nabla_{\dot{D}d}U_{A\underline{\dot{A}}}\right)Y_{\dot{C}\underline{\dot{B}}} + 2i\varepsilon_{ac}U_{A\underline{\dot{A}}}\nabla_{\dot{D}d}Y_{\dot{C}\underline{\dot{B}}} - \\
& - i\varepsilon_{ac}\left(\nabla_{\dot{D}d}U_{A\dot{C}}\right)Y_{\dot{A}\dot{B}} - i\varepsilon_{ac}U_{A\dot{C}}\nabla_{\dot{D}d}Y_{\dot{A}\dot{B}} - \varepsilon_{ac}\varepsilon_{\dot{C}\underline{\dot{B}}}\nabla_{\dot{D}d}\nabla^{B}{}_{\underline{\dot{A}}}W_{AB} + \\
& + i\varepsilon_{\dot{C}\underline{\dot{B}}}\nabla_{\dot{D}d}\nabla_{A\underline{\dot{A}}}X_{ac} - 2\varepsilon_{\dot{C}\underline{\dot{B}}}\left(\nabla_{\dot{D}d}U_{A\underline{\dot{A}}}\right)X_{ac} - \\
& - 2\varepsilon_{\dot{C}\underline{\dot{B}}}U_{A\underline{\dot{A}}}\nabla_{\dot{D}d}X_{ac} \qquad\qquad (C.11) \\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Phi_{B\dot{B}ba} = {} & 2i\varepsilon_{\underline{b}c}\varepsilon_{\dot{B}\dot{C}}\nabla_{\dot{D}d}W_{BA}{}^{A}{}_{\underline{a}} + \frac{2}{3}\varepsilon_{\underline{b}c}\varepsilon_{\dot{B}\dot{C}}\nabla_{\dot{D}d}\Lambda_{B\underline{a}} + 2\varepsilon_{\underline{b}c}\varepsilon_{\dot{B}\dot{C}}\left(\nabla_{\dot{D}d}\rho^{A}_{\underline{a}}\right)W_{AB} - \\
& -2i\varepsilon_{\underline{b}c}\nabla_{\dot{D}d}Y_{\dot{B}\dot{C}B\underline{a}} + 2\varepsilon_{c\underline{b}}\left(\nabla_{\dot{D}d}\rho_{B\underline{a}}\right)Y_{\dot{B}\dot{C}} - 2\varepsilon_{c\underline{b}}\rho_{B\underline{a}}\nabla_{\dot{D}d}Y_{\dot{B}\dot{C}} - \\
& - i\varepsilon_{c\underline{b}}\left(\nabla_{\dot{D}d}\rho_{\dot{C}\underline{a}}\right)U_{B\dot{B}} + i\varepsilon_{c\underline{b}}\rho_{\dot{C}\underline{a}}\nabla_{\dot{D}d}U_{B\dot{B}} + 2i\varepsilon_{c\underline{b}}\left(\nabla_{\dot{D}d}\rho_{\dot{B}\underline{a}}\right)U_{B\dot{C}} - \\
& - 2i\varepsilon_{c\underline{b}}\rho_{\dot{B}\underline{a}}\nabla_{\dot{D}d}U_{B\dot{C}} + 4\varepsilon_{c\underline{b}}\left(\nabla_{\dot{D}d}\rho^{e}_{\dot{C}}\right)\Phi_{B\dot{B}e\underline{a}} - 4\varepsilon_{c\underline{b}}\rho^{e}_{\dot{C}}\nabla_{\dot{D}d}\Phi_{B\dot{B}e\underline{a}} - \\
& - 2\varepsilon_{c\underline{b}}\nabla_{\dot{D}d}\nabla_{B\dot{B}}\rho_{\dot{C}\underline{a}} - 2i\varepsilon_{\dot{B}\dot{C}}\left(\nabla_{\dot{D}d}\rho_{B\underline{b}}\right)X_{c\underline{a}} + 2i\varepsilon_{\dot{B}\dot{C}}\rho_{B\underline{b}}\nabla_{\dot{D}d}X_{c\underline{a}} + \\
& + 2\left(\nabla_{\dot{D}d}\rho_{\dot{C}\underline{b}}\right)\Phi_{B\dot{B}c\underline{a}} - 2\rho_{\dot{C}\underline{b}}\nabla_{\dot{D}d}\Phi_{B\dot{B}c\underline{a}} \qquad\qquad (C.12) \\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}U_{B\dot{B}} = {} & -\nabla_{\dot{D}d}Y_{\dot{B}\dot{C}Bc} - \varepsilon_{\dot{B}\dot{C}}\nabla_{\dot{D}d}W_{AB}{}^{A}{}_{c} + \frac{2}{3}i\varepsilon_{\dot{B}\dot{C}}\nabla_{\dot{D}d}\Lambda_{Bc} \qquad\qquad (C.13) \\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}R = {} & 4i\nabla_{\dot{D}d}\nabla^{B}_{\dot{C}}W_{BA}{}^{A}{}_{c} + 12\left(\nabla_{\dot{D}d}X_{cb}\right)W_{\dot{C}\dot{B}}{}^{\dot{B}b} + 12X_{cb}\nabla_{\dot{D}d}W_{\dot{C}\dot{B}}{}^{\dot{B}b} - \\
& - 2iW^{CB}\nabla_{\dot{D}d}Y_{CB\dot{C}c} - 6i\left(\nabla_{\dot{D}d}W_{\dot{A}\dot{B}\dot{C}c}\right)Y^{\dot{A}\dot{B}} + 6iW_{\dot{A}\dot{B}\dot{C}c}\nabla_{\dot{D}d}Y^{\dot{A}\dot{B}} + \\
& + 12\left(\nabla_{\dot{D}d}U^{B\dot{B}}\right)Y_{\dot{C}\dot{B}Bc} + 12U^{B\dot{B}}\nabla_{\dot{D}d}Y_{\dot{C}\dot{B}Bc} + 4\left(\nabla_{\dot{D}d}U^{B}_{\dot{C}}\right)W_{BC}{}^{C}{}_{c} + \\
& + 4U^{B}_{\dot{C}}\nabla_{\dot{D}d}W_{BC}{}^{C}{}_{c} \qquad\qquad (C.14) \\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}I = {} & 4\nabla_{\dot{D}d}\nabla_{\dot{C}c}R - 48\left(\nabla_{\dot{D}d}X^{ab}\right)\nabla_{\dot{C}c}X_{ab} - 48X^{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}X_{ab} - \\
& - 12W^{AB}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{AB} - 12\left(\nabla_{\dot{D}d}W^{\dot{A}\dot{B}}\right)\nabla_{\dot{C}c}Y_{\dot{A}\dot{B}} - \\
& - 12W^{\dot{A}\dot{B}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}Y_{\dot{A}\dot{B}} - 12\left(\nabla_{\dot{D}d}Y^{\dot{A}\dot{B}}\right)\nabla_{\dot{C}c}W_{\dot{A}\dot{B}} - \\
& - 12Y^{\dot{A}\dot{B}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{A}\dot{B}} - 24\left(\nabla_{\dot{D}d}U^{F\dot{F}}\right)\nabla_{\dot{C}c}U_{F\dot{F}} - \\
& - 24U^{F\dot{F}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}U_{F\dot{F}} + 3P\nabla_{\dot{D}d}\nabla_{\dot{C}c}\overline{P} + 3\left(\nabla_{\dot{D}d}H^{F\dot{F}}\right)\nabla_{\dot{C}c}H_{F\dot{F}} + \\
& + 3H^{F\dot{F}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}H_{F\dot{F}} - 24\left(\nabla_{\dot{D}d}\Phi^{F\dot{F}}_{ab}\right)\nabla_{\dot{C}c}\Phi^{ab}_{F\dot{F}} - \\
& - 24\Phi^{F\dot{F}}_{ab}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Phi^{ab}_{F\dot{F}} - 6\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{F\dot{F}}H_{F\dot{F}} - \\
& - 16i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{Aa}\right)\Lambda_{Aa} - 16i\left(\nabla_{\dot{C}c}\rho^{Aa}\right)\nabla_{\dot{D}d}\Lambda_{Aa} + \\
& + 16i\left(\nabla_{\dot{D}d}\rho^{Aa}\right)\nabla_{\dot{C}c}\Lambda_{Aa} - 16i\rho^{Aa}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Lambda_{Aa} + \\
& + 16i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)\Lambda_{\dot{A}a} + 16i\left(\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)\nabla_{\dot{D}d}\Lambda_{\dot{A}a} - \\
& - 16i\left(\nabla_{\dot{D}d}\rho^{\dot{A}a}\right)\nabla_{\dot{C}c}\Lambda_{\dot{A}a} + 16i\rho^{\dot{A}a}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Lambda_{\dot{A}a} + \\
& + 48\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{Aa}\right)W_{AB}{}^{B}{}_{a} + 48\left(\nabla_{\dot{C}c}\rho^{Aa}\right)\nabla_{\dot{D}d}W_{AB}{}^{B}{}_{a} -
\end{aligned}
$$

$$
\begin{aligned}
&-48\left(\nabla_{\dot{D}d}\rho^{Aa}\right)\nabla_{\dot{C}c}W_{AB}{}^{B}{}_{a}+48\rho^{Aa}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{AB}{}^{B}{}_{a}-\\
&-48\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}-48\left(\nabla_{\dot{C}c}\rho^{\dot{A}a}\right)\nabla_{\dot{D}d}W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}+\\
&+48\left(\nabla_{\dot{D}d}\rho^{\dot{A}a}\right)\nabla_{\dot{C}c}W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}-48\rho^{\dot{A}a}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W_{\dot{A}\dot{B}}{}^{\dot{B}}{}_{a}-\\
&-96iW^{AB}\left(\nabla_{\dot{D}d}\rho_{Aa}\right)\nabla_{\dot{C}c}\rho^{a}_{B}+96iW^{AB}\rho_{Aa}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{a}_{B}-\\
&-96i\left(\nabla_{\dot{D}d}W^{\dot{A}\dot{B}}\right)\rho_{\dot{A}a}\nabla_{\dot{C}c}\rho^{a}_{\dot{B}}-96iW^{\dot{A}\dot{B}}\left(\nabla_{\dot{D}d}\rho_{\dot{A}a}\right)\nabla_{\dot{C}c}\rho^{a}_{\dot{B}}+\\
&+96iW^{\dot{A}\dot{B}}\rho_{\dot{A}a}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{a}_{\dot{B}}-96i\rho_{\dot{A}a}\left(\nabla_{\dot{D}d}\rho^{a}_{\dot{B}}\right)\nabla_{\dot{C}c}W^{\dot{A}\dot{B}}+\\
&+48i\rho_{\dot{A}a}\rho^{a}_{\dot{B}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}W^{\dot{A}\dot{B}}-48\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho_{Aa}\right)\rho^{a}_{\dot{A}}U^{A\dot{A}}-\\
&-48U^{A\dot{A}}\left(\nabla_{\dot{C}c}\rho_{Aa}\right)\nabla_{\dot{D}d}\rho^{a}_{\dot{A}}+48\left(\nabla_{\dot{C}c}\rho_{Aa}\right)\rho^{a}_{\dot{A}}\nabla_{\dot{D}d}U^{A\dot{A}}+\\
&+48\left(\nabla_{\dot{D}d}U^{A\dot{A}}\right)\rho_{Aa}\nabla_{\dot{C}c}\rho^{a}_{\dot{A}}+48U^{A\dot{A}}\left(\nabla_{\dot{D}d}\rho_{Aa}\right)\nabla_{\dot{C}c}\rho^{a}_{\dot{A}}-\\
&-48U^{A\dot{A}}\rho_{Aa}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{a}_{\dot{A}}-48\left(\nabla_{\dot{D}d}\rho_{Aa}\right)\rho^{a}_{\dot{A}}\nabla_{\dot{C}c}U^{A\dot{A}}+\\
&+48\rho_{Aa}\left(\nabla_{\dot{D}d}\rho^{a}_{\dot{A}}\right)\nabla_{\dot{C}c}U^{A\dot{A}}-48\rho_{Aa}\rho^{a}_{\dot{A}}\nabla_{\dot{D}d}\nabla_{\dot{C}c}U^{A\dot{A}}+\\
&+48i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho_{Aa}\right)\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}+48i\left(\nabla_{\dot{C}c}\rho_{Aa}\right)\nabla_{\dot{D}d}\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}-\\
&-48i\left(\nabla_{\dot{D}d}\rho_{Aa}\right)\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}+48i\rho_{Aa}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{\dot{A}}-\\
&-48i\left(\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho_{\dot{A}a}\right)\nabla^{A\dot{A}}\rho^{a}_{A}-48i\left(\nabla_{\dot{C}c}\rho_{\dot{A}a}\right)\nabla_{\dot{D}d}\nabla^{A\dot{A}}\rho^{a}_{A}+\\
&+48i\left(\nabla_{\dot{D}d}\rho_{\dot{A}a}\right)\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{A}-48i\rho_{\dot{A}a}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla^{A\dot{A}}\rho^{a}_{A}+\\
&+96i\left(\nabla_{\dot{D}d}\Phi^{ab}_{A\dot{A}}\right)\rho^{\dot{A}}_{b}\nabla_{\dot{C}c}\rho^{A}_{a}+96i\Phi^{ab}_{A\dot{A}}\left(\nabla_{\dot{D}d}\rho^{\dot{A}}_{b}\right)\nabla_{\dot{C}c}\rho^{A}_{a}-\\
&-96i\Phi^{ab}_{A\dot{A}}\rho^{\dot{A}}_{b}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{A}_{a}+96i\left(\nabla_{\dot{D}d}\Phi^{ab}_{A\dot{A}}\right)\rho^{A}_{b}\nabla_{\dot{C}c}\rho^{\dot{A}}_{a}+\\
&+96i\Phi^{ab}_{A\dot{A}}\left(\nabla_{\dot{D}d}\rho^{A}_{b}\right)\nabla_{\dot{C}c}\rho^{\dot{A}}_{a}-96i\Phi^{ab}_{A\dot{A}}\rho^{A}_{b}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\rho^{\dot{A}}_{a}+\\
&+96i\left(\nabla_{\dot{D}d}\rho^{A}_{a}\right)\rho^{\dot{A}}_{b}\nabla_{\dot{C}c}\Phi^{ab}_{A\dot{A}}-96i\rho^{A}_{a}\left(\nabla_{\dot{D}d}\rho^{\dot{A}}_{b}\right)\nabla_{\dot{C}c}\Phi^{ab}_{A\dot{A}}+\\
&+96i\rho^{A}_{a}\rho^{\dot{A}}_{b}\nabla_{\dot{D}d}\nabla_{\dot{C}c}\Phi^{ab}_{A\dot{A}}\,. \qquad\qquad \text{(C.15)}
\end{aligned}
$$

Because of the $\nabla_{\dot{D}d}\nabla_{\dot{C}c}I$ term in $\nabla_{\dot{D}d}\nabla_{\dot{C}c}\nabla_{\dot{B}b}\Lambda_{\dot{A}a}$, we will also need the following terms:

$$
\begin{aligned}
\nabla_{\dot{D}d}\nabla_{\dot{C}c}\overline{P} = &-8i\nabla_{\dot{D}d}W_{\dot{C}\dot{B}}{}^{\dot{B}}{}_{c}-\frac{4}{3}\nabla_{\dot{D}d}\Lambda_{\dot{C}c}+8\left(\nabla_{\dot{D}d}\rho^{\dot{B}}_{c}\right)W_{\dot{C}\dot{B}}-8\rho^{\dot{B}}_{c}\nabla_{\dot{D}d}W_{\dot{C}\dot{B}}+\\
&+2\left(\nabla_{\dot{D}d}\overline{P}\right)\rho_{\dot{C}c}+2\overline{P}\nabla_{\dot{D}d}\rho_{\dot{C}c}+2\left(\nabla_{\dot{D}d}\rho^{C}_{c}\right)H_{C\dot{C}}-2\rho^{C}_{c}\nabla_{\dot{D}d}H_{C\dot{C}}+\\
&+4i\left(\nabla_{\dot{D}d}\rho^{C}_{c}\right)U_{C\dot{C}}-4i\rho^{C}_{c}\nabla_{\dot{D}d}U_{C\dot{C}}-8\nabla_{\dot{D}d}\nabla_{C\dot{C}}\rho^{C}_{c}+\\
&+8\left(\nabla_{\dot{D}d}\Phi_{C\dot{C}cb}\right)\rho^{Cb}+8\Phi_{C\dot{C}cb}\nabla_{\dot{D}d}\rho^{Cb} \qquad\qquad \text{(C.16)}\\
\nabla_{\dot{D}d}\nabla_{\dot{C}c}H_{B\dot{B}} = &\; 8i\varepsilon_{\dot{C}\dot{B}}\nabla_{\dot{D}d}W_{AB}{}^{A}{}_{c}+\frac{4}{3}\varepsilon_{\dot{C}\dot{B}}\nabla_{\dot{D}d}\Lambda_{Bc}+4\varepsilon_{\dot{C}\dot{B}}W_{BC}\nabla_{\dot{D}d}\rho^{C}_{c}-
\end{aligned}
$$

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$$
\begin{aligned}
& - 4i\varepsilon_{\dot{C}\dot{B}} X_{bc} \nabla_{\dot{D}d} \rho^{b}_{B} + 4i\varepsilon_{\dot{C}\dot{B}} \rho^{b}_{B} \nabla_{\dot{D}d} X_{bc} - 2\varepsilon_{\dot{C}\dot{B}} P \nabla_{\dot{D}d} \rho_{Bc} - \\
& - 2\varepsilon_{\dot{C}\dot{B}} H_{B\dot{A}} \nabla_{\dot{D}d} \rho^{\dot{A}}_{c} + 2\varepsilon_{\dot{C}\dot{B}} \rho^{\dot{A}}_{c} \nabla_{\dot{D}d} H_{B\dot{A}} - 4Y_{\dot{C}\dot{B}} \nabla_{\dot{D}d} \rho_{Bc} + \\
& + 4\rho_{Bc} \nabla_{\dot{D}d} Y_{\dot{C}\dot{B}} - 2iU_{B\dot{B}} \nabla_{\dot{D}d} \rho_{\dot{C}c} + 2i\rho_{\dot{C}c} \nabla_{\dot{D}d} U_{B\dot{B}} + \\
& + 8\Phi_{B\dot{B}bc} \nabla_{\dot{D}d} \rho^{b}_{\dot{C}} - 8\rho^{b}_{\dot{C}} \nabla_{\dot{D}d} \Phi_{B\dot{B}bc} - 4\nabla_{\dot{D}d} \nabla_{B\dot{B}} \rho_{\dot{C}c} + \\
& + 8\nabla_{\dot{D}d} \nabla_{B\dot{C}} \rho_{\dot{B}c} + 8\rho^{b}_{\dot{B}} \nabla_{\dot{D}d} \Phi_{B\dot{C}bc} - 8\Phi_{B\dot{C}bc} \nabla_{\dot{D}d} \rho^{b}_{\dot{B}} \qquad \text{(C.17)}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\dot{D}d} \nabla_{\dot{C}c} \Lambda_{Bb} = {} & 3i\varepsilon_{cb} \nabla_{\dot{D}d} \nabla_{B\dot{B}} W_{\dot{C}}{}^{\dot{B}} + 3i\varepsilon_{cb} \nabla_{\dot{D}d} \nabla^{A}_{\dot{C}} Y_{AB} - 3\nabla_{\dot{D}d} \nabla_{B\dot{C}} X_{cb} + \\
& + 6i \left(\nabla_{\dot{D}d} U_{B\dot{C}} \right) X_{cb} + 6iU_{B\dot{C}} \nabla_{\dot{D}d} X_{cb} \qquad \text{(C.18)} \\
\nabla_{\dot{D}d} \nabla_{\dot{C}c} W_{ABCc} = {} & -\varepsilon_{cb} \nabla_{\dot{D}d} \nabla_{\underline{A}\dot{C}} W_{\underline{B}C} + 2i\varepsilon_{cb} \left(\nabla_{\dot{D}d} U_{\underline{A}\dot{C}} \right) W_{\underline{B}C} - \varepsilon_{cb} \varepsilon_{\underline{B}C} \nabla_{\dot{D}d} \nabla_{\underline{A}}{}^{\dot{B}} Y_{\dot{B}\dot{C}} - \\
& - i\varepsilon_{\underline{B}C} \nabla_{\dot{D}d} \nabla_{\underline{A}\dot{C}} X_{cb} - i\varepsilon_{cb} \left(\nabla_{\dot{D}d} U_{C\dot{C}} \right) W_{AB} - 2\varepsilon_{\underline{B}C} \left(\nabla_{\dot{D}d} U_{\underline{A}\dot{C}} \right) X_{cb} - \\
& - 2\varepsilon_{\underline{B}C} U_{\underline{A}\dot{C}} \nabla_{\dot{D}d} X_{cb} \qquad \text{(C.19)} \\
\nabla_{\dot{D}d} \nabla_{\dot{C}c} Y_{AB} = {} & 2i \nabla_{\dot{D}d} Y_{AB\dot{C}c} \qquad \text{(C.20)} \\
\nabla_{\dot{D}d} \nabla_{\dot{C}c} Y_{\dot{A}\dot{B}} = {} & \frac{2}{3} \varepsilon_{\dot{C}\underline{\dot{B}}} \nabla_{\dot{D}d} \Lambda_{\underline{\dot{A}}c} \qquad \text{(C.21)} \\
\nabla_{\dot{D}d} \nabla_{\dot{C}c} X_{ab} = {} & -\frac{2}{3} i\varepsilon_{c\underline{a}} \nabla_{\dot{D}d} \Lambda_{\dot{C}\underline{b}} \,. \qquad \text{(C.22)}
\end{aligned}
$$

All other differential relations we will need involve combinations of vector and spinor covariant derivatives. Terms containing these expressions are very important for our analysis, since a lot of the derivatives we are looking for come from them. They may be written, using the commutation relations, as vector derivatives of the relations we have seen plus torsion and curvature terms. Some of the torsion terms require differential relations involving undotted and dotted spinorial derivatives which we compute now:

$$
\begin{aligned}
\nabla^{a}_{A} \nabla^{c}_{\dot{C}} \rho^{b}_{\dot{B}} = {} & -\frac{i}{4} \varepsilon_{\dot{C}\dot{B}} \varepsilon^{cb} \nabla^{a}_{A} P - \varepsilon_{\dot{C}\dot{B}} \nabla^{a}_{A} X^{cb} - i\varepsilon^{cb} \nabla^{a}_{A} Y_{\dot{C}\dot{B}} + 2\rho^{c}_{\dot{B}} \nabla^{a}_{A} \rho^{b}_{\dot{C}} - \\
& - 2\rho^{b}_{\dot{C}} \nabla^{a}_{A} \rho^{c}_{\dot{B}} \qquad \text{(C.23)} \\
\nabla^{a}_{A} \nabla^{c}_{\dot{C}} \rho^{b}_{B} = {} & \frac{i}{4} \varepsilon^{cb} \nabla^{a}_{A} H_{B\dot{C}} - \varepsilon^{cb} \nabla^{a}_{A} U_{B\dot{C}} + i\nabla^{a}_{A} \Phi^{bc}_{B\dot{C}} + 2\rho^{c}_{B} \nabla^{a}_{A} \rho^{b}_{\dot{C}} - \\
& - 2\rho^{b}_{\dot{C}} \nabla^{a}_{A} \rho^{c}_{B} \qquad \text{(C.24)} \\
\nabla^{d}_{D} \nabla^{c}_{\dot{C}} H_{B\dot{B}} = {} & 8i\varepsilon_{\dot{C}\dot{B}} \nabla^{d}_{D} W_{BC}{}^{Cc} + \frac{4}{3} \varepsilon_{\dot{C}\dot{B}} \nabla^{d}_{D} \Lambda^{c}_{B} + 4\varepsilon_{\dot{C}\dot{B}} W_{BC} \nabla^{d}_{D} \rho^{Cc} - \\
& - 4\varepsilon_{\dot{C}\dot{B}} \rho^{Cc} \nabla^{d}_{D} W_{BC} + 4i\varepsilon_{\dot{C}\dot{B}} X^{cb} \nabla^{d}_{D} \rho^{C}_{b} - 4i\varepsilon_{\dot{C}\dot{B}} \rho^{C}_{b} \nabla^{d}_{D} X^{cb} - \\
& - 2\varepsilon_{\dot{C}\dot{B}} \rho^{c}_{B} \nabla^{d}_{D} P - 2\varepsilon_{\dot{C}\dot{B}} P \nabla^{d}_{D} \rho^{c}_{B} - 2\varepsilon_{\dot{C}\dot{B}} H_{B\dot{A}} \nabla^{d}_{D} \rho^{\dot{A}c} + \\
& + 2\varepsilon_{\dot{C}\dot{B}} \rho^{\dot{A}c} \nabla^{d}_{D} H_{B\dot{A}} - 4Y_{\dot{C}\dot{B}} \nabla^{d}_{D} \rho^{c}_{B} + 4\rho^{c}_{B} \nabla^{d}_{D} Y_{\dot{C}\dot{B}} - 2iU_{B\dot{B}} \nabla^{d}_{D} \rho^{c}_{\dot{C}} + \\
& + 2i\rho^{c}_{\dot{C}} \nabla^{d}_{D} U_{B\dot{B}} - 8\Phi^{bc}_{B\dot{B}} \nabla^{d}_{D} \rho_{\dot{C}b} + 8\rho_{\dot{C}b} \nabla^{d}_{D} \Phi^{bc}_{B\dot{B}} + 8\Phi^{bc}_{B\dot{C}} \nabla^{d}_{D} \rho_{\dot{B}b} - \\
& - 8\rho_{\dot{B}b} \nabla^{d}_{D} \Phi^{bc}_{B\dot{C}} - 4\nabla^{d}_{D} \nabla_{B\dot{B}} \rho^{c}_{\dot{C}} + 8\nabla^{d}_{D} \nabla_{B\dot{C}} \rho^{c}_{\dot{B}} \qquad \text{(C.25)} \\
\nabla_{Dd} \nabla_{\dot{C}c} U_{B\dot{B}} = {} & -\nabla_{Dd} Y_{\dot{C}\dot{B}Bc} + \varepsilon_{\dot{C}\dot{B}} \nabla_{Dd} W_{BC}{}^{C}{}_{c} - \frac{2}{3} i\varepsilon_{\dot{C}\dot{B}} \nabla_{Dd} \Lambda_{Bc} \qquad \text{(C.26)} \\
\nabla^{d}_{D} \nabla^{c}_{\dot{C}} \Phi^{ab}_{B\dot{B}} = {} & 2i\varepsilon_{\dot{C}\dot{B}} \varepsilon^{c\underline{b}} \nabla^{d}_{D} W_{BC}{}^{C\underline{a}} + \frac{2}{3} \varepsilon_{\dot{C}\dot{B}} \varepsilon^{c\underline{b}} \nabla^{d}_{D} \Lambda^{\underline{a}}_{B} + 2\varepsilon_{\dot{C}\dot{B}} \varepsilon^{c\underline{b}} W_{BC} \nabla^{d}_{D} \rho^{C\underline{a}} - \\
& - 2\varepsilon_{\dot{C}\dot{B}} \varepsilon^{c\underline{b}} \rho^{C\underline{a}} \nabla^{d}_{D} W_{BC} + 2i\varepsilon^{c\underline{b}} \nabla^{d}_{D} Y_{\dot{C}\dot{B}B}{}^{\underline{a}} + 2\varepsilon^{c\underline{b}} Y_{\dot{C}\dot{B}} \nabla^{d}_{D} \rho^{\underline{a}}_{B} -
\end{aligned}
$$

$$
\begin{aligned}
&- 2\varepsilon^{c\underline{b}}\rho^{\underline{a}}_{B}\nabla^d_D Y_{\dot C\dot B} - i\varepsilon^{c\underline{b}}U_{B\dot B}\nabla^d_D\rho^{\underline{a}}_{\dot C} + i\varepsilon^{c\underline{b}}\rho^{\underline{a}}_{\dot C}\nabla^d_D U_{B\dot B} + 2i\varepsilon^{c\underline{b}}U_{B\dot C}\nabla^d_D\rho^{\underline{a}}_{\dot B} - \\
&- 2i\varepsilon^{c\underline{b}}\rho^{\underline{a}}_{\dot B}\nabla^d_D U_{B\dot C} + 2i\varepsilon_{\dot C\dot B}X^{c\underline{b}}\nabla^d_D\rho^{\underline{a}}_{B} - 2i\varepsilon_{\dot C\dot B}\rho^{\underline{a}}_{B}\nabla^d_D X^{c\underline{b}} + 2\Phi^{\underline{b}c}_{B\dot B}\nabla^d_D\rho^{\underline{a}}_{\dot C} - \\
&- 2\rho^{\underline{a}}_{\dot C}\nabla^d_D\Phi^{\underline{b}c}_{B\dot B} - 4\varepsilon^{c\underline{b}}\Phi^{\underline{a}e}_{B\dot B}\nabla^d_D\rho_{\dot Ce} + 4\varepsilon^{c\underline{b}}\rho_{\dot Ce}\nabla^d_D\Phi^{\underline{a}e}_{B\dot B} - 2\varepsilon^{c\underline{b}}\nabla^d_D\nabla_{B\dot B}\rho^{\underline{a}}_{\dot C}\,.
\end{aligned}
$$

All these expressions should be enough for the direct computation of (3.9).

D. Calculation of the derivative terms

In this appendix, we compute the terms arising in the calculation of $\nabla_{\dot Cc}\nabla_{\dot Bb}\nabla_{\dot Aa}\overline{W}^2$ and $\nabla_{\dot Dd}\nabla_{\dot Cc}\nabla_{\dot Bb}\nabla_{\dot Aa}\overline{W}^2$ which contain vector derivatives of the auxiliary fields P, H_m (excluding $\nabla^m H_m$, as justified on the text). For each expression, we indicate its content in terms of the derivatives of interest. The derivatives of some of these expressions will also be necessary; for those expressions, we give their field content in terms of P, H_m. Through all this appendix, we will be only interested in the derivatives we mentioned and, therefore, we will only consider here expressions which contain them. Those expressions we do not consider here simply do not contain such derivatives, as it can be verified using their expansions in appendix C.

D.1 Calculation of the derivative terms in $\nabla_{\dot Cc}\nabla_{\dot Bb}\nabla_{\dot Aa}\overline{W}^2$

From (3.8), the following expressions are necessary:

$$
\nabla^c_{\dot C}\nabla^b_{\dot B}W_{\dot D\dot E} = \frac{i}{4}\varepsilon^{bc}\varepsilon_{\underline{\dot D}\dot B}\varepsilon_{\underline{\dot E}\dot C}\left(P\overline{P} + H^2 - 2\nabla_{A\dot A}H^{A\dot A}\right) + \cdots \tag{D.1}
$$

$$
\nabla^c_{\dot C}\nabla^b_{\dot B}W_{\dot D\dot E\dot A}{}^{a} = -\frac{i}{2}\varepsilon_{\dot B\dot A}\varepsilon^{\underline{a}c}PW_{\dot D\dot E\dot C}{}^{\underline{b}} + \frac{i}{2}\varepsilon_{\dot B\dot A}\varepsilon^{\underline{a}c}H_{E\dot C}Y_{\dot D\dot E}{}^{E\underline{b}} + \cdots \tag{D.2}
$$

$$
\begin{aligned}
\nabla^b_{\dot B}\nabla_{A\dot A}\rho^c_{\dot C} = {} & -\frac{i}{4}\varepsilon^{bc}\varepsilon_{\dot B\dot C}\nabla_{A\dot A}P + \frac{i}{4}\varepsilon^{bc}Y_{\dot A\dot B}H_{A\dot C} + \frac{i}{4}\varepsilon^{bc}\varepsilon_{\dot A\dot B}W_{A}{}^{D}H_{D\dot C} - \\
& - \frac{1}{4}\varepsilon_{\dot A\dot C}\varepsilon^{bc}PU_{A\dot B} + \frac{1}{8}\varepsilon^{bc}\varepsilon_{\dot B\dot C}PU_{A\dot A} - \frac{1}{4}\varepsilon_{\dot A\dot B}X^{bc}H_{A\dot C} + \cdots
\end{aligned} \tag{D.3}
$$

$$
\begin{aligned}
\nabla^c_{\dot C}\nabla_{A\dot A}\rho^{Aa} = {} & \frac{i}{4}\varepsilon^{ca}\nabla_{A\dot A}H^{A}_{\dot C} - \frac{i}{2}\varepsilon^{ca}\overline{P}Y_{\dot A\dot C} + \frac{1}{2}\varepsilon_{\dot A\dot C}\overline{P}X^{ca} - \\
& - \frac{1}{8}\varepsilon^{ca}U_{A\dot A}H^{A}_{\dot C} + \frac{1}{4}\varepsilon^{ca}U_{A\dot C}H^{A}_{\dot A} + \cdots
\end{aligned} \tag{D.4}
$$

$$
\begin{aligned}
\nabla_{A\dot A}\nabla^b_{\dot B}H_{C\dot C} = {} & 2\varepsilon_{\dot C\dot B}\rho^b_C\nabla_{A\dot A}P + 2\varepsilon_{\dot C\dot B}P\nabla_{A\dot A}\rho^b_C + 2\varepsilon_{\dot C\dot B}\rho^{\dot Db}\nabla_{A\dot A}H_{C\dot D} + \\
& + 2\varepsilon_{\dot C\dot B}H_{C\dot D}\nabla_{A\dot A}\rho^{\dot Db} + \cdots\,.
\end{aligned} \tag{D.5}
$$

D.2 Calculation of the derivative terms in $\nabla_{\dot Dd}\nabla_{\dot Cc}\nabla_{\dot Bb}\nabla_{\dot Aa}\overline{W}^2$

In order to compute the derivatives in (3.9), besides the previous expressions we also need

$$
\nabla^d_{\dot D}\nabla^c_{\dot C}\overline{P} = -2i\varepsilon^{dc}\nabla_{C\underline{\dot C}}H^{C}_{\underline{\dot D}} + \cdots \tag{D.6}
$$

$$
\begin{aligned}
\nabla^d_{\dot D}\nabla^c_{\dot C}H_{B\dot B} = {} & 2i\varepsilon^{dc}\varepsilon_{\dot B\underline{\dot C}}\nabla_{B\underline{\dot D}}P - 2\varepsilon_{\dot B\underline{\dot C}}X^{cd}H_{B\underline{\dot D}} - i\varepsilon_{\dot D\dot C}\varepsilon^{dc}W_{B}{}^{C}H_{C\dot B} - \\
& - i\varepsilon_{\dot D\dot C}\varepsilon^{dc}Y_{\dot B}{}^{\dot A}H_{B\dot A} + 4i\varepsilon^{dc}H_{B\dot B}Y_{\dot C\dot D} + \cdots
\end{aligned} \tag{D.7}
$$

$$
\begin{aligned}
\nabla^d_{\dot D}\nabla^c_{\dot C}H_{B\dot B} = {} & -i\varepsilon^{dc}\varepsilon_{\dot C\dot B}\varepsilon_{DB}\nabla_{A\dot A}H^{A\dot A} + i\varepsilon^{dc}\nabla_{B\dot B}H_{D\dot C} - 2i\varepsilon^{dc}\nabla_{B\dot C}H_{D\dot B} + \\
& + \cdots \qquad\qquad \text{(D.8)}\\
\nabla^d_{\dot D}\nabla_{A\dot A}\nabla^c_{\dot C}H_{B\dot B} = {} & i\varepsilon_{\dot C\dot B}\varepsilon_{AB}\varepsilon^{dc}Y_{\dot A\dot D}\nabla_{E\dot E}H^{E\dot E} - i\varepsilon^{dc}Y_{\dot A\dot D}\nabla_{B\dot B}H_{A\dot C} + \\
& + 2i\varepsilon^{dc}Y_{\dot A\dot D}\nabla_{B\dot C}H_{A\dot B} + i\varepsilon_{\dot C\dot B}\varepsilon_{\dot A\dot D}\varepsilon^{dc}W_{AB}\nabla_{E\dot E}H^{E\dot E} - \\
& - i\varepsilon_{\dot A\dot D}\varepsilon^{dc}W_A^{\ C}\nabla_{B\dot B}H_{C\dot C} + 2i\varepsilon_{\dot A\dot D}\varepsilon^{dc}W_A^{\ C}\nabla_{B\dot C}H_{C\dot B} - \\
& - \varepsilon_{\dot C\dot B}\varepsilon_{\dot A\dot D}\varepsilon_{AB}X^{dc}\nabla_{E\dot E}H^{E\dot E} + \varepsilon_{\dot A\dot D}X^{dc}\nabla_{B\dot B}H_{A\dot C} - \\
& - 2\varepsilon_{\dot A\dot D}X^{dc}\nabla_{B\dot C}H_{A\dot B} - \varepsilon_{\dot B\underline{\dot C}}\varepsilon^{dc}U_{A\dot A}\nabla_{B\underline{\dot D}}P - \varepsilon_{\dot C\dot A}\varepsilon^{dc}U_{A\dot D}\nabla_{B\dot B}P + \\
& + 2\varepsilon_{\dot B\dot A}\varepsilon^{dc}U_{A\dot D}\nabla_{B\dot C}P + 2i\varepsilon_{\dot B\underline{\dot C}}\varepsilon^{dc}\nabla_{A\dot A}\nabla_{B\underline{\dot D}}P - \\
& - i\varepsilon_{\dot D\dot C}\varepsilon^{dc}W_B^{\ C}\nabla_{A\dot A}H_{C\dot B} - 2\varepsilon_{\dot B\underline{\dot C}}X^{dc}\nabla_{A\dot A}H_{B\underline{\dot D}} - \\
& - i\varepsilon_{\dot D\dot C}\varepsilon^{dc}Y_{\dot B}^{\ \dot E}\nabla_{A\dot A}H_{B\dot E} + 4i\varepsilon^{dc}Y_{\dot C\dot D}\nabla_{A\dot A}H_{B\dot B} + \cdots \qquad\qquad \text{(D.9)}\\
\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla_{A\dot A}H_{B\dot B} = {} & \nabla^d_{\dot D}\nabla_{A\dot A}\nabla^c_{\dot C}H_{B\dot B} - i\varepsilon_{\dot D\dot B}\varepsilon_{AB}\varepsilon^{dc}Y_{\dot A\dot C}\nabla_{E\dot E}H^{E\dot E} + i\varepsilon^{dc}Y_{\dot A\dot C}\nabla_{B\dot B}H_{A\dot D} - \\
& - 2i\varepsilon^{dc}Y_{\dot A\dot C}\nabla_{A\dot B}H_{B\dot D} - i\varepsilon^{dc}\varepsilon_{\dot D\dot B}\varepsilon_{\dot A\dot C}W_{AB}\nabla_{E\dot E}H^{E\dot E} + \\
& + i\varepsilon^{dc}\varepsilon_{\dot A\dot C}W_A^{\ C}\nabla_{B\dot B}H_{C\dot D} - 2i\varepsilon^{dc}\varepsilon_{\dot A\dot C}W_A^{\ C}\nabla_{C\dot B}H_{B\dot D} - \\
& - \varepsilon_{AB}\varepsilon_{\dot D\dot B}\varepsilon_{\dot A\dot C}X^{dc}\nabla_{E\dot E}H^{E\dot E} + \\
& + \varepsilon_{\dot A\dot C}X^{dc}\nabla_{B\dot B}H_{A\dot D} - 2\varepsilon_{\dot A\dot C}X^{dc}\nabla_{A\dot B}H_{B\dot D} - \\
& - \varepsilon^{dc}\varepsilon_{\dot B\underline{\dot C}}U_{A\dot A}\nabla_{B\underline{\dot D}}P + 2\varepsilon^{dc}\varepsilon_{\dot B\underline{\dot D}}U_{A\dot C}\nabla_{B\underline{\dot A}}P + \cdots . \qquad\qquad \text{(D.10)}
\end{aligned}
$$

Remarkably, when we contract the A, B and $\dot A, \dot B$ indices, lots of terms in the previous equations cancel by themselves. Ignoring the $\nabla_{E\dot E}H^{E\dot E}$ parts, in which we are not interested, we are simply left with

$$
\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla_{A\dot A}H^{A\dot A} = 4\varepsilon^{dc}U^{A}_{\ \underline{\dot C}}\nabla_{A\underline{\dot D}}P + \cdots . \qquad \text{(D.11)}
$$

We also need more derivatives of $\rho^b_{\dot B}$:

$$
\begin{aligned}
\nabla^a_A\nabla^c_{\dot C}\rho^b_{\dot B} = {} & -\frac{i}{2}\varepsilon^{cb}\varepsilon_{\dot C\dot B}\rho^a_A P - \frac{i}{2}\varepsilon^{cb}H_{A\underline{\dot B}}\rho_{\underline{\dot C}a} - \frac{i}{2}\varepsilon_{\dot C\dot B}\varepsilon^{a\underline{b}}H_{A\dot A}\rho^{\dot A\underline{c}} - \\
& - \frac{i}{2}\varepsilon_{\dot C\dot B}\varepsilon^{cb}H_{A\dot A}\rho^{\dot A a} + \cdots \qquad\qquad \text{(D.12)}\\
\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla_{A\dot A}\rho^b_{\dot B} = {} & -\frac{i}{2}\varepsilon^{db}\varepsilon_{\dot D\dot C}\rho^c_{\dot B}\nabla_{A\dot A}P + \frac{i}{2}\varepsilon^{dc}\varepsilon_{\dot D\dot B}\rho^b_{\dot C}\nabla_{A\dot A}P + \cdots \qquad\qquad \text{(D.13)}\\
\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla_{A\dot A}\rho^b_{\dot B} = {} & -i\varepsilon^{\underline{c}b}\varepsilon_{\dot D\dot C}\rho^{\underline{c}}_{\dot B}\nabla_{A\dot A}P - \frac{i}{2}\varepsilon^{cb}\varepsilon_{\dot D\dot C}\rho^{\dot B d}\nabla_{A\dot A}H_{B\dot B} - \\
& - \frac{i}{2}\varepsilon^{dc}\rho^b_{\dot C}\nabla_{A\dot A}H_{B\dot D} + \cdots . \qquad\qquad \text{(D.14)}
\end{aligned}
$$

With respect to derivatives of $\Phi^{ab}_{A\dot B}$, we need:

$$
\nabla^d_{\dot D}\nabla^c_{\dot C}\Phi^{ab}_{B\dot B} = \frac{i}{2}\varepsilon_{\dot C\dot B}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_B^{\ A}H_{A\dot D} + \frac{i}{2}\varepsilon_{\dot D\dot B}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_B^{\ A}H_{A\dot C} + \frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot B\dot C}H_{B\dot D} -
$$

$$
\begin{aligned}
&-\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot B\dot D}H_{B\dot C}-\frac{1}{2}\varepsilon_{\dot C\dot B}\varepsilon^{d\underline{b}}X^{c\underline{a}}H_{B\dot D}-\frac{1}{2}\varepsilon_{\dot D\dot B}\varepsilon^{c\underline{b}}X^{d\underline{a}}H_{B\dot C}-\\
&-i\varepsilon^{c\underline{a}}\varepsilon_{\dot D\dot C}\Phi^{d\underline{b}}_{B\dot B}P-\frac{i}{2}\varepsilon^{d\underline{a}}\varepsilon_{\dot D\dot C}\Phi^{c\underline{b}}_{B\dot B}P+\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}\varepsilon_{\dot D\dot C}\nabla_{B\dot B}P+\cdots \qquad \text{(D.15)}
\end{aligned}
$$

$$
\nabla^d_D\nabla^c_{\dot C}\Phi^{ab}_{B\dot B}=\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}\nabla_{B\dot B}H_{D\dot C}+\cdots \qquad \text{(D.16)}
$$

$$
\begin{aligned}
\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla_{A\dot A}\Phi^{ab}_{B\dot B}=\;&\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}\varepsilon_{\dot D\dot C}\nabla_{A\dot A}\nabla_{B\dot B}P-\\
&-i\varepsilon^{c\underline{a}}\varepsilon_{\dot D\dot C}\Phi^{d\underline{b}}_{B\dot B}\nabla_{A\dot A}P-\frac{i}{2}\varepsilon^{d\underline{a}}\varepsilon_{\dot D\dot C}\Phi^{c\underline{b}}_{B\dot B}\nabla_{A\dot A}P-\\
&-\frac{i}{2}\varepsilon_{\dot C\dot B}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_B{}^D\nabla_{A\dot A}H_{D\dot D}+\frac{i}{2}\varepsilon_{\dot D\dot B}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_B{}^D\nabla_{A\dot A}H_{D\dot C}-\\
&-\frac{i}{2}\varepsilon_{\dot C\dot A}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_A{}^D\nabla_{B\dot B}H_{D\dot D}+\frac{i}{2}\varepsilon_{\dot D\dot A}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}W_A{}^D\nabla_{B\dot B}H_{D\dot C}+\\
&+\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot B\dot C}\nabla_{A\dot A}H_{B\dot D}-\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot B\dot D}\nabla_{A\dot A}H_{B\dot C}-\\
&-\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot A\dot C}\nabla_{B\dot B}H_{A\dot D}+\frac{i}{2}\varepsilon^{c\underline{a}}\varepsilon^{d\underline{b}}Y_{\dot A\dot D}\nabla_{B\dot B}H_{A\dot C}-\\
&-\frac{1}{2}\varepsilon_{\dot C\dot B}\varepsilon^{d\underline{b}}X^{c\underline{a}}\nabla_{A\dot A}H_{B\dot D}-\frac{1}{2}\varepsilon_{\dot D\dot B}\varepsilon^{c\underline{b}}X^{d\underline{a}}\nabla_{A\dot A}H_{B\dot C}-\\
&-\frac{1}{2}\varepsilon_{\dot C\dot A}\varepsilon^{d\underline{b}}X^{c\underline{a}}\nabla_{B\dot B}H_{A\dot D}+\frac{1}{2}\varepsilon_{\dot D\dot A}\varepsilon^{c\underline{b}}X^{d\underline{a}}\nabla_{B\dot B}H_{A\dot C}+\cdots . \qquad \text{(D.17)}
\end{aligned}
$$

We did not include in (D.15) and (D.17) those terms with a $U_{A\dot A}\nabla_{B\dot B}P$ factor, simply because these terms exist in partial calculations, but overall they cancel. The term with $\nabla_{A\dot A}\nabla_{B\dot B}P$ does not cancel in (D.17), but this expression appears in (C.6) as $\nabla^d_{\dot D}\nabla^c_{\dot C}\nabla^B{}_{\underline{\dot A}}$ $\Phi^{ab}_{B\underline{\dot B}}$. Therefore, this term appears as $\nabla^B{}_{\underline{\dot A}}\nabla_{B\underline{\dot B}}P$, a commutator that does not have vector derivatives.

Replacing all the appropriate expressions in (C.15), with suitable contraction or symmetrization of the adequate indices, and adding all terms, we are led to the surprising and exciting result that $\nabla_{\dot D d}\nabla_{\dot C c}I$, like $\nabla_{\dot C c}I$, also has no derivatives of $P, H_{A\dot A}$ other than $\nabla_{A\dot A}H^{A\dot A}$. We analyze the other terms from (3.9) in the main text.

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